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Credit gap risk in a first passage time model with jumps

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The payoff of many credit derivatives depends on the level of credit spreads. In particular, credit derivatives with a leverage component are subject to gap risk, a risk associated with the occurrence of jumps in the underlying credit default swap spreads. In the framework of first passage time models, we consider a model that addresses these issues. The principal idea is to model a credit quality process as an Itô integral with respect to a Brownian motion with a stochastic volatility. Using a representation of the credit quality process as a time-changed Brownian motion, one can derive formulas for conditional default probabilities and credit spreads. An example for a stochastic volatility process is the square root of a Lévy-driven Ornstein-Uhlenbeck process. The model can be implemented efficiently using a technique called Panjer recursion. Calibration to a wide range of dynamics is supported. We illustrate the effectiveness of the model by valuing a leveraged credit-linked note.

Keywords: gap risk; credit spreads; credit dynamics; first passage time models; stochastic volatility; general Ornstein-Uhlenbeck processes

1. Introduction

Other than being subject exclusively to default risk, the payoff of many credit derivatives is determined explicitly by the level of CDS spreads, the spreads of credit default swaps. In this case, the dynamics of CDS spreads play a significant role in product valuation. Examples of such products are default swaptions and credit derivatives with a leverage component. The latter are in addition sensitive to *gap risk*, a risk that is linked to the occurrence of jumps in the evolution of credit spreads, even if such jumps do not lead to default. An example of such a product is the *leveraged credit-linked note*, which is described in detail later.

Empirical research indicates that credit spreads are subject to sudden and unexpected jumps. Furthermore, CDS spreads exhibit frequent positive jumps, which are attributed to the arrival of bad news, and which typically affect CDS spreads of all maturities, see Schneider *et al.* (2010).

The model that we propose captures these stylised facts and is suitable for valuing products that are exposed to gap risk, while at the same time maintaining a high degree of analytic

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tractability.

There are generally two approaches to modelling credit risk: the structural and the reduced-form approach. In reduced-form models, default is not linked to economic variables, but is an unpredictable Poisson-type event, and the main object of the modeller's attention is the hazard rate of the jump process describing default. This approach has been overwhelmingly popular with practitioners, its main advantage being its tractability: It is generally straightforward to fit a given term structure of CDS spreads and the techniques are very similar to those of interest rate modelling. The literature on this type of models is vast, the papers by Jarrow and Turnbull (1995), Lando (1998) and Duffie and Singleton (1999) are only a few classic examples.

From the point of view of spread dynamics, modelling the default time as a totally unpredictable stopping time is not entirely satisfactory. Even with a low initial hazard rate, such a model will assign a non-negligible probability to the possibility of the credit defaulting without a prior movement in the credit spread. Defaults of this type are very uncommon in practice. The default swap market is efficient enough that default events are almost always preceded by a significant widening of credit spreads. It is this spread widening that is the real jump event that market participants need to worry about.

The ability of a model to assign probability mass to spread widening scenarios is constrained by the probability assigned to defaults in low spread scenarios, as the model must fit the initial credit spread. From a practical point of view, failure to assign enough probability to spread widening scenarios can lead to a dangerous underpricing of credit spread gap risk. The phenomenology of default that we are trying to capture is the following: a credit with a low default swap spread does not default "out of the blue", but rather some kind of regime change takes place, causing the credit spread to widen, after which the credit may either default or eventually recover.

We implement this idea via a first passage time model where a *credit quality process* exhibits stochastic volatility. More explicitly, the volatility process is a Levy-driven Ornstein-Uhlenbeck process. A jump in the volatility process is the "regime switch" we alluded to earlier.

The current trend is to interpret the class of structural models in a wide sense to include any model where default is modelled as the first hitting time of a certain threshold by an abstract observable credit quality process. In this sense our model is structural, though the term "first passage time model" is technically more accurate. The structural approach, pioneered by Merton (1974), has been developed by Black and Cox (1976), Longstaff and Schwartz (1995), and many others.

As the name suggests, in a first passage time model, the computation of default probabilities is equivalent to computing the distributions of first passage times. The simplest case is that of a Brownian motion hitting a constant barrier, where a simple closed-form solution exists. This simple set-up however does not allow one to fit a given term structure of credit spreads. Furthermore, in reality, credit spreads exhibit strong jump dynamics. Several extensions where the credit quality process is modelled by a jump-diffusion or a Lévy process were brought forward to overcome these problems, e.g. Zhou (2001), Kiesel and Scherer (2007), Baxter (2007) and Cariboni and Schoutens (2007). However, in all of these cases, computing first passage times is intractable or computationally very demanding.

Overbeck and Schmidt (2005) propose a simple solution to the problem of calibrating a first passage time model to a term structure of credit spreads by considering the first hitting time of a time-changed Brownian motion to a constant barrier. The time change is continuous, strictly increasing and deterministic. Because both the time change and the underlying Brownian motion are continuous, one can easily adapt the analytic formula from the simple Brownian case and obtain an analytic calibration to a term structure of default probabilities.

It turns out that one can extend the model to incorporate jumps while maintaining much of the tractability of the Overbeck-Schmidt-model. The principal idea is to incorporate a *stochastic* time change, which is still continuous, but is chosen such that resulting credit spreads incorporate jumps. We consider a credit quality process X to be a time-changed Brownian motion with a stochastic, continuous and strictly increasing time change independent of the Brownian motion. The credit quality process X can also be represented as an Itô integral $X = \int_0^{\cdot} \sigma_u dW_u$ with a Brownian motion W and a volatility process σ. Our standard example for the volatility process σ is the square root of an Ornstein-Uhlenbeck process driven by a compound Poisson process. In the time-changed Brownian motion interpretation, we can write X as $X_t = B_{\Lambda_t}$ with B a Brownian motion and $\Lambda_t = \int_0^t \sigma_u^2 du$. Because the time change Λ is continuous and independent of B, we retain all the tractability of the deterministic case. However, a salient feature of the model is that despite the continuity of the credit quality process and the time change, jumps in the volatility σ induce jumps in credit spreads.

Aside from introducing the model, we consider its implementation, calibration and the valuation of gap risk. Given the state of the credit quality process and its volatility, one can efficiently compute the whole term structure of default probabilities or credit spreads using a technique called Panjer recursion, Panjer (1981). Implementation then boils down to a combination of Monte Carlo simulation for determining the state of the credit quality process and numerical computation of term structures. Under suitable choices for the dynamics parameters the error of calibration to a term structure of CDS spreads is as small as 10^{-6} basis points, while at the same time the model may be calibrated to a wide range of dynamical behaviours. Even though the CDS spread in the first passage time model vanishes as maturity of the underlying CDS is approached, thereby excluding short-term default events, one may approximate short-term default events by including the possibility of large jumps in the volatility.

The paper is structured as follows: in Section 2 we introduce the term structure of CDS spreads and the leveraged credit-linked note. In Section 3, we introduce the first passage time model with jumps and consider the Overbeck-Schmidt model as a special case. Furthermore, we derive formulas for conditional default probabilities, from which credit spreads may be determined, and we establish that jumps in the stochastic volatility of the credit quality process translate into jumps in CDS spreads. In Section 4, we derive the algorithm for efficiently computing term structures of default probabilities and credit spreads. Stylised properties of credit spread term structures in the model and calibration to implied term structures are treated in Section 5. Since there are currently no liquid market instruments to calibrate the dynamics, we analyse the spread distribution and demonstrate the range of viable dynamics that may be obtained in the model. In Section 6 we apply the model to the valuation of a leveraged credit-linked note. Finally, we conclude in Section 7.

2. Credit derivatives

2.1. Notation

Throughout, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ be a complete probability space endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$, representing the information available in the market. In particular, $\mathcal{F}_0$ is $\mathbf{P}$-trivial. We assume that $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual hypotheses, i.e., $\mathcal{F}_0$ contains all $\mathbf{P}$-null sets of $\mathcal{F}$ and $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous. We also assume that the probability space is rich enough to support any objects that we define. If not otherwise stated, all processes are $(\mathcal{F}_t)_{t\geq 0}$-adapted.

We assume that $\mathbf{P}$ is a risk-neutral measure, that is, $\mathbf{P}$ is a probability measure equivalent to the real-world probability measure and such that discounted prices are $\mathbf{P}$-martingales. It follows that the market is free of arbitrage.

2.2. Credit default swaps

The fundamental product of the credit derivatives market is the *credit default swap (CDS)*. Given an underlying entity, such as a company, it is a contract between two counterparties, the protection buyer and the protection seller, that insures the protection buyer against the loss incurred by default of the underlying entity within a fixed time interval. The protection buyer regularly pays a constant premium, the *credit spread* or *CDS spread*, which is fixed at inception, up until maturity of the CDS or the default event, whichever occurs first. This stream of payments is termed the *premium leg* of the CDS. In return, the protection seller agrees to compensate the protection buyer for the loss incurred by default of the underlying entity at the time of default in case this occurs before maturity. This constitutes the *protection leg* of the CDS. The CDS spread that makes the value of the premium leg and the protection leg equal is the *fair CDS spread*.

More precisely, let $r \in \mathbb{R}_+$ denote the default-free interest rate, assumed to be constant for simplicity. Furthermore, assume that the payment at default is a fraction $(1-R)$ of the notational amount, $R \in [0,1)$. Denote by τ the random time of the default event. In our setup, τ is an $(\mathcal{F}_t)_{t\geq 0}$-stopping time. The distribution function of τ conditional on the information flow $(\mathcal{F}_t)_{t\geq 0}$ is denoted by $P(t,T) := \mathbf{P}(\tau \leq T|\mathcal{F}_t)$. Denote by $s(t,T)$ the fair credit spread at time t of a CDS with maturity T. Entering into a CDS involves no initial cash-flow, that is, the market value of a CDS at inception is 0;[1] in other words, the discounted fair values of the premium and the default legs are equal. From these considerations, one can show that, on $\{\tau > t\}$, the *fair credit spread* or *fair CDS spread* $s(t,T)$ at time t is given by

$$\frac{s(t,T)}{1-R} = \frac{\int_t^T \mathbf{e}^{-r(u-t)}\,\mathrm{d}P(t,u)}{\int_t^T \mathbf{e}^{-r(u-t)}\,(1-P(t,u))\,\mathrm{d}u}. \tag{1}$$

On $\{\tau \leq t\}$ or for $t \geq T$, we set $s(t,T) = 0$. The mapping $T \mapsto s(t,T)$ is the *term structure of credit spreads* at time t. Given a term structure of CDS spreads $s(t,T)$, $T \geq t$, one can infer the risk-neutral default probabilities $\mathbf{P}(t,T)$, $T \geq t$, from Equation (1), and vice versa.

The *mark-to-market value* of an existing CDS position is expressed as the cost of unwinding the transaction by entering into an offsetting CDS position. Assume a CDS contract with maturity T entered at time $v \leq t$ from the point of view of the protection seller. On $\{\tau > t\}$, the value of the position at time t is

$$\begin{aligned} V_t &= s(v,T)\int_t^T \mathbf{e}^{-r(u-t)}\,(1-P(t,u))\,\mathrm{d}u - (1-R)\int_t^T \mathbf{e}^{-r(u-t)}\,P(t,\mathrm{d}u) \\ &= (s(v,T)-s(t,T))\int_t^T \mathbf{e}^{-r(u-t)}\,(1-P(t,u))\,\mathrm{d}u. \end{aligned} \tag{2}$$

If default occurs prior to T, that is, $\tau < T$, we set $V_\tau = -(1-R)$.

[1] Since 2009, following an initiative of the International Swaps and Derivatives Association to facilitate netting, credit default swaps are nowadays entered into with standardised premiums of 25, 100, 500 or 1000 basis points. As a consequence, entering into a CDS normally requires an upfront premium to be paid. In the analysis of our model, we neglect the changes in market quoting conventions, since the qualitative results on the spread dynamics are not affected.

2.3. Shape and dynamics of the term structure

The term structure of credit spreads has been extensively studied.[2] Let us outline some stylised facts about the shape and dynamics of the term structure. A wide variety of term structure shapes has been observed in the market, such as upward sloping, flat, hump-shaped and downward sloping curve, see for example Helwege and Turner (1999), Zhou (2001), Fons (1994) and Helwege and Turner (1999).

Another common observation is that short-term credit spreads do not tend to zero as maturity tends to zero, but are significantly greater than zero, see e.g. Duffie and Lando (2001), Zhou (2001), (Duffie and Singleton 2003, Ch. 3) and (Lando 2004, Ch. 2). This indicates that, for any time to maturity, market participants presume a positive probability of unexpected and instantaneous default.

There is also a significant amount of research that indicates that credit spreads are subject to jumps, i.e., in addition to continuous behaviour of credit spreads through time, credit spreads may change by sudden and unexpected jumps, see Johannes (2000), Zhou (2001), Das (2002), Dai and Singleton (2003), Tauchen and Zhou (2006), Zhang *et al.* (2009). Schneider *et al.* (2010) infer the following empirical stylised facts for CDS spreads:

- A jump affects broad ranges of the CDS maturity spectrum. This is economically motivated by the fact that unfavourable events usually affect contracts of both short and long maturities, and similarly, when expectations about the overall credit quality change, the entire term structure of CDS spreads reacts.
- Jumps in CDS spreads are mostly positive. The arrival of bad news such as financial distress causes sudden upward moves in CDS spreads, because protection sellers demand higher compensation for bearing higher risk. Good news, on the other hand, tend to propagate gradually.
- The one-year CDS spread exhibits time-series variation different from CDS spreads of higher maturities.

2.4. Leveraged credit-linked note

Let us now consider a credit derivative whose payoff is sensitive to the occurrence of jumps in credit spreads. A *leveraged credit-linked note* is particularly sensitive to jumps in CDS spreads, even if a jump does not lead to default.[1] The principal idea is that an investor sells protection on an amount of default risk that is a multiple k, the *leverage factor*, of his investment amount. The motivation for taking leveraged exposure is to earn a certain multiple $\tilde{k}$ of the credit spread. Most likely, his investment will not suffice to compensate the loss incurred by default. Therefore, a trigger is agreed to terminate the structure while the cost of closing the position is still likely to be sufficiently covered by the investment amount. The cost of closing the position depends on the level of credit spreads, hence the investor is exposed mainly to spread risk and to default risk only to a lesser extent.

In more detail, the issuer structures the note as follows (see Figure 1): For simplicity, assume an investment amount of 1, which is deposited in a default-free account earning the risk-free rate r. In addition, protection is sold by entering a fair CDS with notional $k > 1$ earning a

[2]The term credit spread also refers to the *yield spread*, which is the yield difference of defaultable and default-free zero-coupon bonds of the same maturity. There are some subtle differences between yield spreads and CDS spreads, mainly due to factors such as liquidity of the underlying and restrictions regarding short-selling. However, we assume that stylised facts of the yield spread term structure that can be related to the credit risk component of the underlying entity apply to the CDS term structure as well.

[1]A *credit-linked note (CLN)* is a note or bond paying an enhanced coupon to an investor for bearing the credit risk of a reference entity; see (Bielecki and Rutkowski 2002, Section 1.3.3) for a general description.

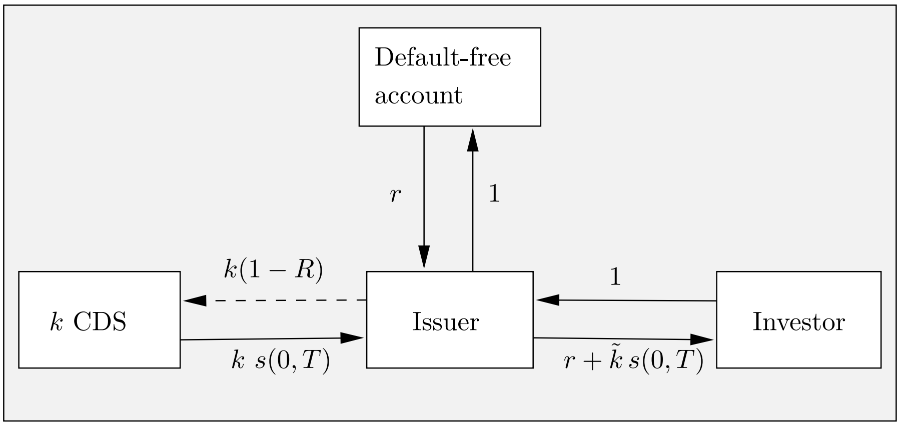

Figure 1. Leveraged credit-linked note with leverage factor k and notional 1. Cash flows at inception and while the note is alive.

spread of $k\,s(0,T)$. The investor receives a fixed coupon until either maturity of the note or until a trigger event takes place. The size of the coupon is $r+\tilde{k}\,s(0,T)$ with $\tilde{k}\,s(0,T)$, $\tilde{k}\le k$, the premium associated with the note, which is generated by the CDS position. The trigger event is defined as follows: denote by V_t^k the mark-to-market value at time t of the underlying CDS position with notional k. The trigger event takes place at time $S=\inf\{t\in(0,T]:V_t^k\le -K\}$, with $0<K\le 1$ a pre-defined trigger level. At S, the note is unwound by withdrawing the investment amount 1 from the deposit account and by closing the CDS position, i.e., by entering the offsetting position, at a cost of $-V_S^k$. Observe that possibly $V_S^k<-1$, in which case the issuer must cover the missing amount required to unwind the CDS position. For this type of risk, called *gap risk*, the issuer is compensated with a premium of $(k-\tilde{k})\,s(0,T)$. In the case where $V_S^k\ge -1$, the investor receives the remainder of the structure, $1+V_S^k$. Given K, valuation of the note essentially means determining the fair factor $\tilde{k}$.

Clearly, the trigger time S depends on the evolution of the underlying CDS spread, cf. Equation (2). Furthermore, the amount of the redemption payment $\max(1+V_S^k,0)$ is undetermined until S. Assuming a model in which CDS spreads evolve continuously, the mark-to-market value V^k evolves continuously as well. Unless a default takes place, the trigger time is $S=\inf\{t\in(0,T]:V_t^k=-K\}$ and $V_S^k\ge -1$. Hence, a gap event takes place only when default happens without a prior trigger event. On the contrary, assuming a model in which CDS spreads are subject to jumps, upward jumps in CDS spreads translate into downward jumps in the mark-to-market value of the CDS, and possibly $V_S^k<-1$, so the issuer faces gap risk even when no default takes place.

In some models, we can determine the fair factor $\tilde{k}$ in a model-free way by employing no-arbitrage arguments. Assume first that CDS spreads evolve continuously through time, so V^k evolves continuously as well. Moreover, assume that there is no risk of an unpredictable jump-to-default (i.e., there is no default “totally out of the blue”). In this case, the trigger time is $S=\inf\{t\in(0,T]:V_t^k=-K\}$ and there is no gap risk at all, and the fair factor is $\tilde{k}=k$. Now suppose that CDS spreads are constant, so the note is exposed to default risk only (in which case $V_t^k=0$, since $s(t,T)=s(0,T)$, for all $t\in[0,T)$). The trigger time then coincides with the default time, in which case the investor loses his invested capital. The payoff of this position is equivalent to the payoff of a short position in $1/(1-R)$ CDS, so $\tilde{k}=1/(1-R)$.

Regardless of any model assumption, we can infer upper and lower bounds for the factor $\tilde{k}$. The upper bound is k as the note's spread pickup is funded by the underlying CDS position. To determine the lower bound, observe that the investor in the leveraged note is exposed to default risk and additionally to spread risk. An investor in a CDS position with notional $1/(1-R)$ is exposed to the same loss in case of default, but may terminate the investment at the same trigger time S with a smaller loss. Hence, $\tilde{k} > 1/(1-R)$.

To determine the factor $\tilde{k}$, consider the cash flows to the note issuer discounted to time 0 (observe that the cash flows to the issuer isolate the gap risk component). The risk-neutral value of these cash flows is given by

$$\begin{aligned} V_0^{\text{gap}} &= \mathbb{E}\left((k-\tilde{k})s(0,T) \int_0^T \mathrm{e}^{-ru} \mathbf{1}_{\{S>u\}} \, \mathrm{d}u - \mathrm{e}^{-rS} \max(-V_S^k - 1, 0) \right) \\ &= (k-\tilde{k})s(0,T) \int_0^T \mathrm{e}^{-ru} \mathbf{P}(S>u) \, \mathrm{d}u \\ &\quad - \int_{(0,T]\times(1,\infty)} \mathrm{e}^{-ru} \, (x-1) \, \mathbf{P}(S \in \mathrm{d}u, -V_S^k \in \mathrm{d}x). \end{aligned} \tag{3}$$

The fair gap risk spread is obtained by setting $V_0^{\text{gap}} = 0$, that is,

$$(k-\tilde{k})s(0,T) = \frac{\int_{(0,T]\times(1,\infty)} \mathrm{e}^{-ru} \, (x-1) \, \mathbf{P}(S \in \mathrm{d}u, -V_S^k \in \mathrm{d}x)}{\int_0^T \mathrm{e}^{-ru} \, \mathbf{P}(S>u) \, \mathrm{d}u}. \tag{4}$$

The gap risk component in the valuation formula (3) is an option with payoff $\max(-V_S^k - 1, 0)$ at time S, which the note issuer sells to the investor. The option premium is the stream of payments $\int_0^T (k-\tilde{k})s(0,T)\mathbf{1}_{\{S>u\}} \, \mathrm{d}u$ that is earned while the note is alive, so that Equation (3) may be interpreted as the valuation formula for a gap option.

Clearly, to value a gap option requires a model that includes jumps in the evolution of credit spreads.

3. First passage time model with jumps

We introduce the model and derive a closed formula for conditional default probabilities. We then introduce the special case where the variance σ^2 of the credit quality process is a Lévy-driven Ornstein-Uhlenbeck process.

3.1. Credit quality process with stochastic volatility

Definition 3.1: The *credit quality process* $X = (X_t)_{t\geq 0}$ of a risky entity is defined to be

$$X_t = \int_0^t \sigma_s \, \mathrm{d}W_s, \quad t \geq 0,$$

where W is an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion and σ is a strictly positive $(\mathcal{F}_t)_{t\geq 0}$-adapted càdlàg process independent of W with $\mathbf{P}(\int_0^t \sigma_s^2 \, \mathrm{d}s < \infty) = 1$, $t \geq 0$, and $\lim_{t\to\infty} \int_0^t \sigma_s^2 \, \mathrm{d}s = \infty$ $\mathbf{P}$–a.s..[1]

[1]The requirement $\lim_{t\to\infty} \int_0^t \sigma_s^2 \, \mathrm{d}s = \infty$ $\mathbf{P}$–a.s. ensures that $\tau < \infty$ $\mathbf{P}$–a.s., as will become clear later on.

The default time τ of the risky entity associated with the credit quality process X is the first time that X hits a barrier $b < 0$:

$$\tau = \inf\{t \geq 0 : X_t \leq b\}.$$

Denote the quadratic variation process of X by $\Lambda = (\Lambda_t)_{t\geq 0}$, with $\Lambda_t = \int_0^t \sigma_s^2 \, \mathrm{d}s$. Observe that Λ is continuous, strictly increasing and $(\mathcal{F}_t)_{t\geq 0}$-adapted.

The credit quality process can be expressed as a time-changed Brownian motion: Define the family of $(\mathcal{F}_t)_{t\geq 0}$-stopping times $\tau_t = \inf\{s \geq 0 : \Lambda_s > t\}$, $t \geq 0$. By application of the Theorem of Dambis, Dubins-Schwarz (see e.g. (Karatzas and Shreve 1998, Chapter 3.4.B)) the process B, with $B_t = X_{\tau_t}$, $t \geq 0$, is an $(\mathcal{F}_{\tau_t})$-Brownian motion. Conversely, the credit quality process X can be expressed as a time-changed Brownian motion, $X_t = B_{\Lambda_t}$, $t \geq 0$. We refer to B as the DDS-Brownian motion of X. Furthermore, for every $t > 0$, Λ_t is an $(\mathcal{F}_{\tau_t})_{t\geq 0}$-stopping time. Since Λ is continuous, strictly increasing and $\lim_{t\to\infty} \Lambda_t = \infty$ $\mathbf{P}$–a.s., it follows that $\tau_{\Lambda_t} = \Lambda_{\tau_t} = t$.

Example 3.2 Deterministic time change, Overbeck-Schmidt model**:** In the model proposed by Overbeck and Schmidt (2005), the time change Λ is chosen to be deterministic. Assume given the distribution $F(t) = \mathbf{P}(\tau \leq t)$, $t \geq 0$, of the default time (e.g. derived from market credit spreads). If the time-change Λ is given by

$$\Lambda_t = \left(\frac{b}{\mathrm{N}^{(-1)}\left(\frac{F(t)}{2}\right)} \right)^2, \quad t \geq 0, \tag{5}$$

where $\mathrm{N}^{(-1)}$ denotes the inverse of the Normal distribution function, then τ admits the distribution F. Furthermore, if the distribution of τ admits a density, then the time-change is absolutely continuous, and $\Lambda_t = \int_0^t \sigma_s^2 \, \mathrm{d}s$, with σ a nonnegative square-integrable function, the volatility of X. There exists a representation of X as a stochastic integral $X_t = \int_0^t \sigma_s \, \mathrm{d}W_s$, for some Brownian motion W (cf. Theorem 3.4.2 of Karatzas and Shreve (1998)). The volatility σ can be interpreted as the *default speed* in the sense that the higher the default speed, the higher the likelihood of crossing the default barrier.

It is easily shown that for $t \leq T$, and on $\{\tau > t\}$, the probability of default until T conditional on $\mathcal{F}_t$ is given by $P(t,T) = 2\mathrm{N}\left(\frac{b - X_t}{\sqrt{\Lambda_T - \Lambda_t}}\right)$, where N denotes the standard normal distribution function. Note that the dynamics of $(P(t,T))_{t\geq 0}$ in the Overbeck-Schmidt-model are fully determined by calibration to market-given default probabilities; it is not possible to assign different dynamics to the same term structure of default probabilities. Furthermore, the process $(P(t,T))_{t\geq 0}$ is continuous; in other words, it is impossible to generate jumps in a default probability process when the time change is deterministic and continuous. It is thus clear that the model is unsuitable for valuing gap risk.

3.2. Conditional default probabilities

We state a formula for conditional default probabilities. The key for establishing the formula lies in the independence of the DDS-Brownian motion of X and its quadratic variation (the time-change associated with the DDS-Brownian motion). This is is proved in Proposition C.2 in Appendix C. All other proofs are in Appendix B.

Proposition 3.3: *Let X be a credit quality process with volatility process σ. Let $\tau = \inf\{t \geq 0 : X_t \leq b\}$ be the associated default time. On $\{\tau > t\}$, the probability of default until time $T > t$, conditional on $\mathcal{F}_t$, is given by*

$$\mathbf{P}(\tau \leq T|\mathcal{F}_t) = \mathbb{E}\left(2\mathrm{N}\left(\frac{b - X_t}{\sqrt{\Lambda_T - \Lambda_t}}\right)\Big|\mathcal{F}_t\right) \quad \mathbf{P}\text{–}a.s.. \tag{6}$$

Furthermore, if (X, σ) has the Markov property, then, on $\{\tau > t\}$, the conditional default distribution is

$$\mathbf{P}(\tau \leq T|\mathcal{F}_t) = \mathbf{P}(\tau \leq T|X_t, \sigma_t) = \mathbb{E}\left(2\mathrm{N}\left(\frac{b - X_t}{\sqrt{\Lambda_T - \Lambda_t}}\right)\Big|X_t, \sigma_t\right) \quad \mathbf{P}\text{–}a.s.. \tag{7}$$

Setting $t = 0$, we obtain a formula for unconditional default probabilities:

$$\mathbf{P}(\tau \leq T) = 2\mathbb{E}\left(\mathrm{N}\left(\frac{b}{\sqrt{\Lambda_T}}\right)\right), \quad T \geq 0. \tag{8}$$

3.3. Variance as Lévy-driven Ornstein-Uhlenbeck process

We put the model to work by specifying the variance process σ^2 to be a mean-reverting process with jumps. Candidates as drivers for the variance process are Lévy processes: they incorporate jumps, and we can build Markov processes by specifying the dynamics of the variance with respect to Lévy processes, see (Protter 2005, Theorem V.32).

For our modelling purpose, it is sufficient to consider variance processes driven by compound Poisson processes, where jumps are rare events – the economic rationale being that jumps in CDS spreads are triggered by the arrival of "bad news" in the market. Nonetheless, the statements in this section also apply to infinite activity processes.

As an explicit example we model the variance process as a *Lévy-driven Ornstein-Uhlenbeck process* (LOU process). If it is driven by a pure-jump process with positive jumps, an LOU process moves up by jumps and decays exponentially in-between the jumps. Models where an asset price's variance is driven by an LOU process were first considered by Barndorff-Nielsen and Shephard (2001). For details on LOU processes, see also Norberg (2004), (Schoutens 2003, Chapter 5) and (Cont and Tankov 2004, Chapter 15.3.3). Let us specify the credit-quality process model with the variance driven by an LOU process.

Proposition 3.4: *Let W be an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion, and let Z be an $(\mathcal{F}_t)_{t\geq 0}$-subordinator (that is, a Lévy process with nondecreasing paths) independent of W. Furthermore, let $a \in \mathbb{R}_+$ and let θ be a bounded and càdlàg function, such that σ^2 (defined below) is strictly positive. The stochastic process $X = (X_t)_{t\geq 0}$,*

$$X_t = \int_0^t \sigma_s \, \mathrm{d}W_s, \quad t \geq 0, \tag{9}$$

with σ^2 the solution of

$$\mathrm{d}\sigma_t^2 = a(\theta(t) - \sigma_{t-}^2)\, \mathrm{d}t + \mathrm{d}Z_t, \tag{10}$$

is a credit quality process in the sense of Definition 3.1. Moreover, (X, σ) is a Markov process with respect to $(\mathcal{F}_t)_{t\geq 0}$.

If Z is a compound Poisson process, the variance and the time-change increment are given by

$$\sigma_t^2 = \mathbf{e}^{-at}\,\sigma_0^2 + \int_0^t \mathbf{e}^{-a(t-u)}\,a\,\theta(u)\,\mathrm{d}u + \sum_{0<u\leq t} \mathbf{e}^{-a(t-u)}\,\Delta Z_u \tag{11}$$

$$\begin{aligned}\Lambda_T - \Lambda_t = \int_t^T \sigma_u^2\,\mathrm{d}u &= \left(1-\mathbf{e}^{-a(T-t)}\right)\frac{\sigma_t^2}{a}\\ &+ \int_t^T \theta(u)\left(1-\mathbf{e}^{-a(T-u)}\right)\mathrm{d}u + \frac{1}{a}\sum_{t<u\leq T}\left(1-\mathbf{e}^{-a(T-u)}\right)\Delta Z_u.\end{aligned} \tag{12}$$

Equation (11) is verified by applying the Itô formula to $\mathbf{e}^{at}\,\sigma_t^2$ (which establishes *a fortiori* that a solution to Equation (10) exists). The time-change increment is obtained by integrating each term of Equation (11). A sample path of σ^2 and of $\Lambda = \int_0^t \sigma_s^2\,\mathrm{d}s$ is given in Figure 2. Furthermore, the 5-year default probability and corresponding term hazard rate (as a proxy for CDS spreads, cf. Appendix A) are given.

3.4. Jumps in default probabilities and credit spreads

The continuity of the credit quality process X and the associated time-change Λ are essential to derive the formula for conditional default probabilities, Equation (6), from which credit spreads can be computed. It turns out that for a credit quality process with variance driven by an LOU process, jumps in the variance process propagate to default probabilities and credit spreads.

Proposition 3.5: *Let X be a credit quality process with LOU variance process σ^2 as in Proposition 3.4 with σ^2 driven by a subordinator. Let $\tau = \inf\{t > 0 : X_t \leq b\}$ be the associated default time. Fix $T > 0$ and let $(P(t,T))_{t\leq T}$ be the associated conditional default probability process. Then, for $\mathbf{P}$-almost all $\omega \in \{\tau > t\}$, $(P(t,T))_{t\leq T}$ is a process whose jumps are positive and*

$$\Delta\sigma_t^2(\omega) = 0 \iff \Delta P(t,T)(\omega) = 0, \qquad \textit{for all } T > t.$$

A straightforward consequence of this Proposition is that a jump of σ_t^2 triggers a jump in all conditional default probability processes $(P(t,T))_{t\leq T}$, $T \geq 0$.

Essentially, the claim follows by observing that conditional default probabilities (cf. Equation (6)) are conditional expectations involving the time-change increment $\Lambda_T - \Lambda_t$, and the *increment* $\Lambda_T - \Lambda_t$ jumps if σ_t^2 does, although Λ itself is a continuous process. In the special case where σ^2 is driven by a compound Poisson process this is also readily observed by considering the representation (12). The full proof is given in the appendix.

For CDS spreads we have the following result:

Proposition 3.6: *Let $(s(t,T))_{0\leq t\leq T}$ be the CDS spread process for maturity T. Then $(s(t,T))_{0\leq t\leq T}$ is càdlàg, and for $t \leq T$ and for $\mathbf{P}$-almost all $\omega \in \{\tau > t\}$,*

$$\Delta s(t,T)(\omega) > 0 \iff (\Delta P(t,u)(\omega) > 0, \textit{ for some } u \in (t,T]).$$

Similarly, let $(s(t,t+T))_{t\geq 0}$ be the CDS spread process for time-to-maturity T. Then $(s(t,t+T))_{t\geq 0}$ is càdlàg and for $t \leq T$ and $\mathbf{P}$–a.a. $\omega \in \{\tau > t\}$,

$$\Delta s(t,t+T)(\omega) > 0 \iff \Delta P(t,u)(\omega) > 0, \textit{ for some } u \in (t,T+t].$$

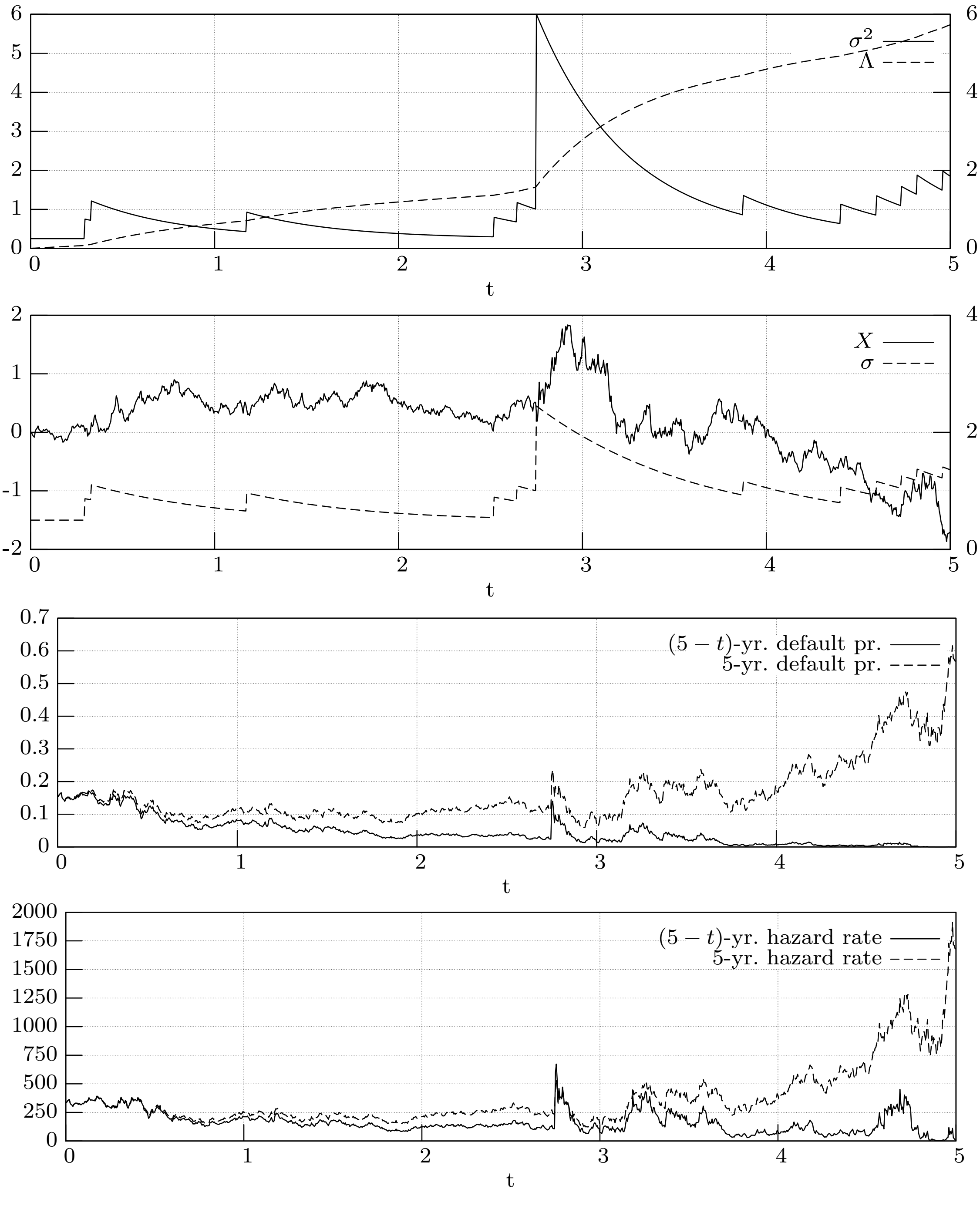

Figure 2. Example of variance process and credit quality process. Top: variance process σ^2 (left axis); time-change Λ (right axis). Second from top: credit quality process X (left axis); volatility σ (right axis). Second from bottom: 5-year default probability process, with decaying time-to-maturity (continuous line) and with fixed time-to-maturity (dashed line). Bottom: Corresponding term hazard rate. Parameters: $a = 2$, $\theta \equiv 0.25$, $\sigma_0^2 = 0.25$; σ^2 is driven by a compound Poisson process with jump intensity $\lambda = 2$ and discrete jump size distribution with jump sizes $0.05, 5$ with probabilities $0.95, 0.05$, respectively. The barrier is $b = -3$.

An example of a jump in the variance process and its propagation to default probabilities and credit spreads is given in Figure 2.

Obviously, the model excludes events where credit spreads jump for selected maturities only. However, this is compatible with the observation that credit spreads tend to jump together.

Finally, it is easily seen that a jump in the variance process cannot lead to default $\mathbf{P}$–a.s.. It suffices to observe that $\tau = \inf\{t > 0 : X_t \leq b\}$ is a predictable stopping time, whereas the jumps of the driving compound Poisson process are totally inaccessible. However, we shall see later that jump-to-default events can be approximated by large jumps in the variance.

4. Computation of default probabilities and credit spreads

The implementation of the LOU variance model is a combination of Monte Carlo simulation and analytical computation. On $\{\tau > t\}$, and conditional on X_t, σ_t^2, default probabilities $P(t,T) = \mathbf{P}(\tau \leq T | X_t, \sigma_t^2)$, $T > t$, can be computed numerically, so that Monte Carlo simulation reduces to simulating X_t and σ_t^2. The advantage of such an algorithm is that valuation of a product involving $P(t,T)$ or $s(t,T)$ requires simulation only until t instead of T. For example, to value a default swaption one needs to simulate merely until option expiry instead of maturity of the underlying CDS.

4.1. Jump size distribution of time-change $\boldsymbol{\Lambda}$

Assume the credit quality process model (X, σ^2) of Proposition 3.4, with σ^2 an LOU process driven by a compound Poisson process Z. We wish to compute conditional default probabilities $\mathbf{P}(\tau \leq T | X_t, \sigma_t)$, $0 \leq t \leq T$, according to Equation (7). Inspection of the formula for the time-change increments $\Lambda_T - \Lambda_t$, Equation (12), reveals that computation of the conditional expectation (7) essentially entails computing the distribution of

$$L_{t,T} := \sum_{t<u\leq T} \left(1 - \mathbf{e}^{-a(t-u)}\right) \Delta Z_u.$$

Let Z have jump intensity λ and jump size $Y > 0$. For every $t \leq T$, the random variable $L_{t,T}$ follows a compound Poisson distribution (see (Sato 1999, Chapter 22) or Norberg (2004)),

$$L_{t,T} \sim \mathrm{CPO}(\lambda(T-t), (1 - \mathbf{e}^{-a(T-S)})Y), \tag{13}$$

with S uniformly distributed on $(t,T]$, i.e., $S \sim U(t,T)$, and independent of Y. Moreover, $L_{t,T} \stackrel{\mathcal{L}}{=} L_{0,T-t}$, hence it suffices to compute the distributions of $L_T := L_{0,T}$. The following result states the distribution of the compounding variate $(1 - \mathbf{e}^{a(T-S)})\, Y$ of L_T.

Lemma 4.1: *For $T > 0$, let $S \sim U(0,T)$ and let Y be a $\mathbf{P}$–a.s. strictly positive random variable independent of S. The distribution of $\left(1 - \mathbf{e}^{a(T-S)}\right) Y$ is given by*

$$F(x) = \mathbb{E}\left(-\frac{\ln(1-x/Y)}{aT} \mathbf{1}_{\{[0,1-\mathbf{e}^{-aT})\}}(x/Y)\right) + \mathbf{P}\left(Y \leq \frac{x}{1-\mathbf{e}^{-aT}}\right), \quad x \in \mathbb{R}. \tag{14}$$

number of simulations	CPU time (seconds)	MSE (at $t = 5$)
1000	164.28	0.9369
2000	333.70	0.2725
5000	1966.54	0.0589
10000	8554.59	0.0403

Table 1. Monte Carlo simulation vs. Panjer recursion. CPU time (Panjer recursion): 331.33 CPU secs.

4.2. Panjer recursion

The distribution of L_T can be computed efficiently using *Panjer recursion*, see Panjer (1981) or (McNeil *et al.* 2005, Chapter 10). This method is based on a recursive evaluation formula for a family of compound distributions. In our implementation it has proven to be numerically more stable to assume a discrete distribution of the compounding variate, although the distribution function of the compounding variate in Equation (14) is continuous. For the compound Poisson case, the method works as follows: Suppose N is a Poisson distributed random variable with intensity λ and let the compounding variate Y take values in the nonnegative integers. Set $f(i) = \mathbf{P}(Y = i)$, $i = 1, 2, \ldots$. For a random variable $L \sim \text{CPO}(\lambda, Y)$, its distribution $g(i) = \mathbf{P}(L = i)$, $i = 1, 2, \ldots$, is given by

$$g(i) = \sum_{n=0}^{i} f^{n*}(i)\mathbf{P}(N = n), \quad i = 1, 2, \ldots,$$

where $f^{n*}(i)$ denotes the n-fold convolution product of f at i. The number of computations required for determining $g(i)$ is of the order i^2. The result by Panjer states that

$$g(i) = \frac{\lambda}{i} \sum_{j=1}^{i} j\, f(j)\, g(i - j), \quad i = 1, 2, \ldots,$$

in which case the number of computations required for determining $g(i)$ is of the order i. By proper scaling on an equidistant grid, the method can also be used for discrete nonnegative compounding variates not restricted to integers.

To illustrate the pickup in computational speed using Panjer recursion, we compare the computation of the distributions of Λ_{t_i}, with $t_i = i/10$, $i = 0, \ldots, 200$, for points $x = (x_i)_{i=0,\ldots,8000}$, using Monte Carlo simulation and Panjer recursion. For the Monte Carlo simulation, Λ_t was simulated at 200 time points with 8000 grid points each, with 1000, 2000, 5000 and 10000 simulations. The CPU times of the simulations are given in Table 1. In addition, the simulation results are compared with the computation using Panjer recursion by considering the simulation mean square error (MSE) relative to the value obtained by Panjer recursion.

4.3. Algorithm

Suppose we wish to compute default probabilities $P(u_j, u_j + t_i)$, $j = 1, \ldots J$, $i = 1, \ldots, N$.[1] The full simulation algorithm, outlined below, is given in pseudo-code in Algorithm **1**. For each t_i,

[1]For notational simplicity we compute an $r \times n$ matrix of default probabilities; other setups of time points and time-to-maturities are possible.

we compute the distribution of L_{t_i} on an equidistant space grid $x_1, \ldots, x_M$. Next, we simulate K paths of (σ, X), yielding $(\sigma_{u_j})^k_{j=1,\ldots,J}$ and $(X_{u_j})^k_{j=1,\ldots,J}$, $k = 1, \ldots, K$. For each u_j, we check if default has occurred. However, simulating on a discrete time grid underestimates the occurrence of the default event. Hence, in addition we sample an indicator variable that determines whether default has occurred between two time points. This is realised by applying a well-known result to determine the barrier hitting event of a Brownian bridge. Taking into account that X is a Brownian motion with a continuous time-change, the indicator takes value 1 with probability $\mathbf{e}^{-2(b-X_{u_{j-1}})(b-X_{u_j})/(\Lambda_{u_j}-\Lambda_{u_{j-1}})}$, $j = 2, \ldots J$, cf. (Glasserman 2004, Section 6.4). For each time point u_j and time-to-maturity t_i, we determine $\Lambda_{t_i+u_j} - \Lambda_{u_j}$ by computing the deterministic part and then the expectation using the distribution L_{t_i}.[2] In this way we obtain a term structure of default probabilities, which serves as the basis for computing credit spreads according to Equation (1). In the algorithm we use the credit triangle (see Appendix A) to compute credit spreads.

Require: $t_1 = 0 < \ldots < t_N$ // time grid
Require: $x_1 = 0 < \ldots < x_M$ // space grid
Require: $u_1, \ldots, u_J$ // desired maturities
Require: K // number of simulations
Require: b // default barrier
Require: a, θ, λ, σ_0^2, F // volatility process parameters, F jump size distribution
1: // Panjer recursion
2: **for** $i = 1$ to N **do**
3: **for** $j = 1$ to J **do**
4: compute $\mathbf{P}(L_{t_i} \in [x_{j-1}, x_j))$
5: **end for**
6: **end for**
7: // simulation step
8: **for** $k = 1$ to K **do**
9: $\tau^k \leftarrow \infty$ // default time of k-th simulation
10: **for** $j = 1$ to J **do**
11: simulate $\sigma^k_{u_j}$ and $X^k_{u_j}$
12: sample $d \leftarrow \mathbf{1}_{\{\min_{u_{j-1}<s\leq u_j} X_s \leq b\}}$ cond. on $X_{u_{j-1}}$ and X_{u_j} // (see text)
13: **if** $d = 1$ or $X^k_{u_j} \leq b$ **then**
14: $\tau^k \leftarrow u_j$
15: $P^k(u_j, u_j + t_i) \leftarrow 1$, $s^k(u_j, t_i) \leftarrow 0$ // for all $i = 1, \ldots, N$
16: **next** k // exit k-th simulation
17: **end if**
18: **for** $i = 1$ to N **do**
19: $h \leftarrow (1 - \mathbf{e}^{-at_i})\, \sigma^2_{u_j}/a + \int_{u_j}^{u_j+t_i} \theta(r) \left(1 - \mathbf{e}^{-a(u_j+t_i-r)}\right) \mathrm{d}r$
20: $P^k(u_j, u_j + t_i) \leftarrow 2 \sum_{m=1}^M \mathrm{N}\left(\frac{b-X_{u_j}}{\sqrt{h+x_{m-1}/a}}\right) \mathbf{P}\left(L_{t_i} \in [x_{m-1}, x_m)\right)$
21: $s^k(u_j, u_j + t_i) \leftarrow -(1 - R) \ln(1 - P(u_j, u_j + t_i))/t_i$ // credit triangle
22: **end for**
23: **end for**
24: **end for**

Algorithm 1: Computation of conditional default probabilities

[2] Actually, the deterministic part need not be computed for every simulation, hence for efficiency the computation of Line 19 should take place outside the loop $k = 1, \ldots, K$.

5. Stylised properties and calibration of the model

We are now in a position to further study the properties of the model. We shall examine attainable credit-spread term structures, calibration to market data and the spread distribution.

5.1. Shape of credit-spread term structure

Examples of credit spread term structures with different parameters are given in Figure 3. Recall from the stylised empirical facts stated in Section 2.3 that a term structure of credit spreads may assume different shapes. Typically, an investment grade company's term structure is upward sloping, reflecting lower default risk in the near future compared to higher uncertainty in the long term. A speculative-grade company may have an inverted term structure, indicating that the firm faces higher short-term default risk, but is more likely to survive in the long-term conditional on survival in the short-term.

A common observation is that credit spreads are strictly positive as time-to-maturity tends to zero, indicating that default may happen suddenly and unexpectedly. In Section 3.4 we established that the model is not capable of producing this property - credit spreads vanish as time-to-maturity tends to zero, and the default time is predictable. However, the possibility of large jumps in the variance may allow for "near-jump-to-default" events (see also case (d) of the examples presented in Section 5.3). We would then expect the spread term structure to be very steep at the short end.

By choosing extreme values for either the barrier or the initial variance, we obtain sharply humped term structures that approximate inverted term structures. Both cases reflect a low credit quality: default becomes more likely as either the credit quality process approaches the barrier or as the variance increases.

5.2. Calibration to a term structure of default probabilities

Calibration is the process of assigning parameters of the model such that the model reproduces market prices such as the term structure of credit spreads. Further market prices, such as prices of default swaptions, provided they are available and liquid, may be suitable for calibrating the dynamics. In the absence of a liquid market for such claims, calibrating the dynamics via historical data may be a feasible alternative. We focus here on calibration to a given term structure and illustrate the attainable range of dynamics in the following subsection by calculating the spread distribution.

In the LOU model of Proposition 3.4, the deterministic function θ and the initial variance σ_0^2 will be chosen to reproduce a given term structure. The remaining parameters – mean reversion constant a, jump intensity λ, jump size distribution F, barrier b – are chosen to determine the dynamics. It should be noted, however, that the deterministic function θ influences the dynamics and that the parameters for the dynamics influence the calibration of θ. It is also the case that calibration to a given term structure imposes some restrictions on the dynamics parameters – in other words, given a set of dynamics parameters, it is not possible to achieve satisfactory calibration to an arbitrary term structure; this is outlined in detail below. The overall calibration process is to assign parameters for the dynamics first and then to calibrate to the spot curve.

The allocation of the parameters to spot curve calibration and dynamics calibration is justified as follows: in a model with a jump intensity of zero, the resulting time-change process is deterministic, which corresponds to the Overbeck-Schmidt model. In this case, the only parameters that are relevant for calibration to a given spot curve are the initial variance σ_0^2 and the deterministic function θ, and the dynamics are fixed by the deterministic time-change. Only when

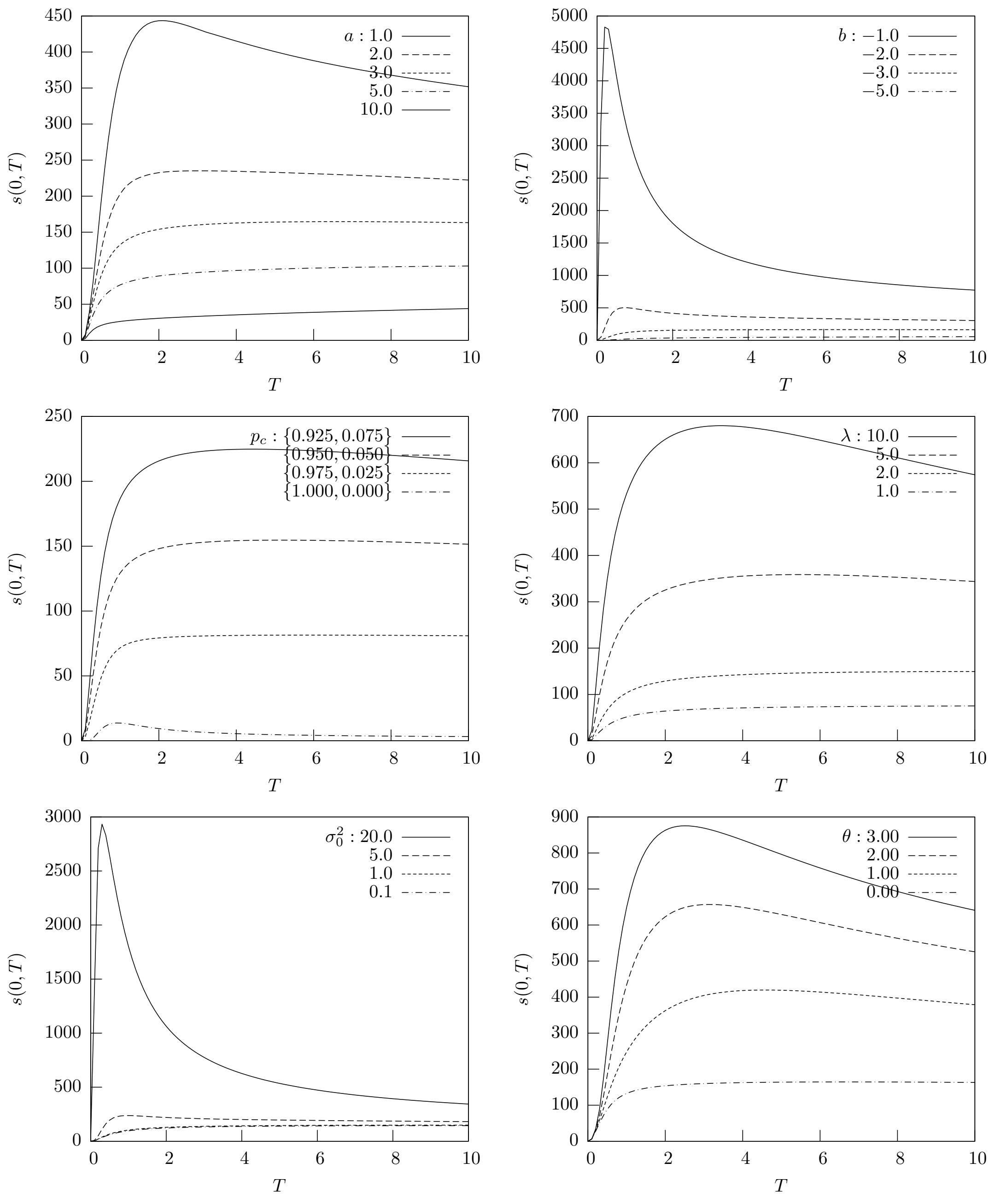

Figure 3. Impact of individual parameters on credit spread term structure. The standard parameter set is $a = 3$, $b = -3$, $\lambda = 2$, $\sigma_0^2 = 3$, $\theta \equiv 0$, jump size in $\{20, 0.1\}$ with probabilities $\{0.05, 0.95\}$. p_c denotes the jump size probabilities. In each case, the curves are ordered from top to bottom at $T = 10$.

the jump intensity is greater than zero do the dynamics change, in which case all parameters allocated to the dynamics calibration become relevant for the dynamics.

Assume given a set of default probabilities $P(T_i) := \mathbf{P}(\tau \leq T_i)$, $T_1 < \cdots < T_n$, derived from market-given credit spreads (together with a recovery assumption). For fixed mean reversion a, barrier b, jump intensity λ and jump size distribution F, the objective is to determine σ_0^2

and θ to match the given default probabilities. Since default probabilities in the LOU-model are expectations, cf. Equation (6), there is in general no analytic method to calibrate. Moreover, it turns out that it is not even guaranteed that an exact solution of the calibration problem exists. We require that $P(t,T)$, $T \geq t$, be strictly increasing, capturing the fact that a risky entity may default at any time, for every $t \geq 0$, $\mathbf{P}$–a.s.. Clearly, by inspection of Equation (6), this condition is met if the time-change Λ is strictly increasing $\mathbf{P}$–a.s., or, equivalently, if $\sigma_t^2 > 0$, $t \geq 0$, $\mathbf{P}$–a.s..

Although an exact solution to the calibration problem with a certain set of dynamics parameters may not exist, satisfactory calibration quality to a given term-structure may always be obtained. Indeed, a model without jump component is equivalent to the OS-model, where analytic and exact calibration is possible. By choosing suitably moderate jump dynamics, an arbitrary calibration quality may be achieved, as we shall see below.

We calibrate numerically by minimising the error between market-given and model-computed default probabilities. In the following, we shall always assume θ to be piecewise constant,

$$\theta(t) = \sum_{i=1}^{n} \theta(T_i)\mathbf{1}_{(T_{i-1},T_i]}(t), \quad t > 0, \tag{15}$$

with the convention $T_0 = 0$. We define the *root mean square error (RMSE)* between market default probabilities and model default probabilities as

$$\delta(\sigma_0^2, \theta; P, a, b, \lambda, F) := \sqrt{\sum_{i=1}^{n} \frac{T_i - T_{i-1}}{T_n} \left(P(T_i) - 2\mathbb{E}\mathrm{N}\left(b/\sqrt{\Lambda_{T_i}} \right) \right)^2}, \tag{16}$$

where the expectation denotes the model-given default probability for maturity T_i, cf. Equation (8) and Λ_{T_i} is given by (cf. Equation (12))

$$\Lambda_{T_i} = \left(1 - \mathbf{e}^{-a\,T_i}\right) \frac{\sigma_0^2}{a} + \sum_{j=1}^{i} \theta(j) \left[T_j - T_{j-1} - \frac{\mathbf{e}^{-a(t-T_j)} - \mathbf{e}^{-a(t-T_{j-1})}}{a} \right] + \frac{1}{a} \sum_{0<u\leq T_i} \left(1 - \mathbf{e}^{-a(T_i - u)}\right) \Delta Z_u. \tag{17}$$

In order for $P(t,T)$, $T \geq t$, to be strictly increasing for all t, we require that (cf. Equation (11))

$$\sigma_t^2 = \mathbf{e}^{-at}\,\sigma_0^2 + \sum_{i=1}^{n} \theta(T_i)\,\mathbf{e}^{-at} \left(\mathbf{e}^{a(t\wedge T_i)} - \mathbf{e}^{a(t\wedge T_{i-1})} \right) + \sum_{0<u\leq t} \mathbf{e}^{-a(t-u)}\,\Delta Z_u > 0, \quad t \geq 0.$$

Taking into account that jumps are positive, the condition is satisfied if θ satisfies

$$\theta(T_i) > -\frac{\sigma_0^2 + \sum_{j=1}^{i-1} \theta(T_j) \left(\mathbf{e}^{aT_j} - \mathbf{e}^{aT_{j-1}} \right)}{\mathbf{e}^{aT_i} - \mathbf{e}^{aT_{i-1}}}, \quad i = 1, \ldots, n. \tag{18}$$

Define the set

$$\Theta = \left\{ (\theta(T_1), \ldots, \theta(T_n)) \in \mathbb{R}^n : (\theta(T_1), \ldots, \theta(T_n)) \text{ satisfies (18)} \right\}.$$

For the model-given probabilities to be well-defined requires additionally that $\lambda \geq 0$, $a > 0$, $b \leq 0$ and $F(0) = 0$. Under these conditions, the solution to the calibration problem is then given by

$$(\sigma_0^{\star 2}, \theta^\star(T_1), \dots, \theta^\star(T_n)) := \underset{\{\sigma_0^2 \in \mathbb{R}_+, (\theta(T_1),\dots,\theta(T_n)) \in \Theta\}}{\arg\min} \delta(\sigma_0^2, \theta; P, a, b, \lambda, F). \tag{19}$$

Example 5.1 : Assume given default probabilities $P(T_i) = 1 - \mathbf{e}^{-hT_i}$, at times $T_i = i$ (years), $i = 1, \dots, 10$, with $h = 0.03$. We calibrate the model to these default probabilities, for different jump size distributions, jump intensities λ and mean reversion constants a.

The distribution of L_{0,T_i}, $i = 1, \dots, 10$, is computed on an equidistant grid of 5000 points in the interval $[0, 120]$. The barrier is $b = -3$. The mean reversion a and the jump intensity λ are both chosen to take different parameters between 1 and 10 and the following jump size distributions are considered:

(i) The jump size is 1/4.
(ii) The jump size is 1/2.
(iii) The jump size distribution is exponential with parameter $\nu = 4$, i.e., $F(x) = 1 - \mathbf{e}^{-\nu x}$.
(iv) The jump size is 0.1 with probability 0.95 and 20 with probability 0.05. Here, we enlarged the grid for computing the distributions of $(L_{0,T_i})_{i=1,\dots,10}$ to 11000 points on the interval $[0, 264]$.

The RMSE's for credit spreads (in basis points, 1bp = 0.01%) was computed from market-given default probabilities by

$$s(T_i) = 10^4 \cdot (1 - R) \frac{P(T_i)}{\sum_{j=1}^{i} (1 - P(T_j))(T_j - T_{j-1})}, \quad i = 1, \dots, n.$$

The recovery rate was $R = 0.4$. In all examples, for small default intensities λ and high mean reversion a the RMSE between market-given and model-given credit spreads is of the order 10^{-6}, whereas for large default intensities and small mean reversion the order of the error can be as large as 100. This is explained as follows: Increasing the frequency of jumps, increases the volatility of the credit quality process, which in turn increases the likelihood of the credit quality process hitting the barrier. A too high jump intensity may thus inhibit satisfactory calibration to a given term structure. To understand the effect of a low mean reversion, one must study Equation (12). First of all, the jump size in the time-change Λ is scaled by the mean reversion. Secondly, a low mean reversion "dampens" the function θ at the short end of the term structure, which is then amplified for longer time-to-maturity. Thus, for low mean reversion, calibrating the short term default probabilities selects high values for θ, whose effect is intensified for long term default probabilities. It follows that under many circumstances a low mean reversion fails to calibrate well to either the short end or the long end of a given term structure.

5.3. Spread distribution and dynamics

To understand the dynamic behaviour of credit spreads, we consider the spread distribution as implied by the model. For this we first calculate the distribution of conditional default probabilities and then determine the term hazard rate as a proxy for credit spreads (see Appendix A). We give examples that demonstrate the range of attainable dynamics.

Fixing $t \geq 0$ and $T \geq t$, let us first calculate the distribution of condition default probabilities

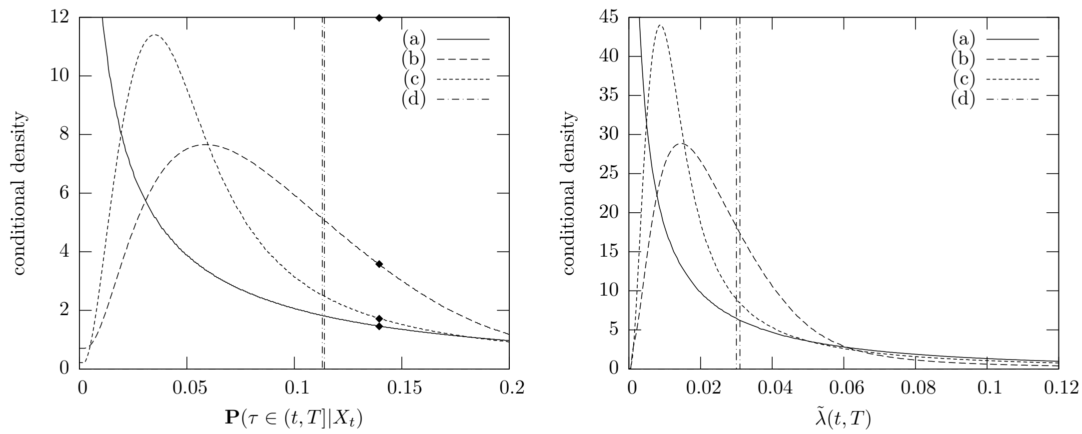

Figure 4. Distributions of $P(t,T)$ (left) and $\tilde{\lambda}(t,T)$ (right) conditional on $\{\tau > t\}$ for different dynamics. The term hazard rate $\tilde{\lambda}(t,T)$ is an approximation of the credit spread, i.e., $s(t,T) \approx (1-R)\tilde{\lambda}(t,T) \cdot 10^4$ (in basis points). We choose $t = 1, T = 5$ and an initial hazard rate of 3%. The diamonds mark the initial 5-year default probability. The parameters are given in Table 2.

$P(t,T)$. Define

$$f_{\sigma_t^2}(X_t) := 2\mathbb{E}\left(\mathrm{N}\left(\frac{b - X_t}{\sqrt{\Lambda_T - \Lambda_t}}\right)\Big| X_t, \sigma_t^2\right).$$

Observe that $f_{\sigma_t^2}$ is continuous and strictly decreasing so that its inverse $f_{\sigma_t^2}^{(-1)}$ exists.

Proposition 5.2: *For $T \geq 0$ and $t \leq T$, the distribution function of $P(t,T)$ conditional on $\{\tau > t\}$ is*

$$\mathbf{P}(P(t,T) \leq x | \tau > t)$$
$$= \mathbb{E}\left[\mathrm{N}\left(\frac{-f_{\sigma_t^2}^{(-1)}(x)}{\sqrt{\Lambda_t}}\right) - \mathrm{N}\left(\frac{2b - f_{\sigma_t^2}^{(-1)}(x)}{\sqrt{\Lambda_t}}\right)\right] \Big/ \left[1 - 2\,\mathbb{E}\left(\mathrm{N}\left(\frac{b}{\sqrt{\Lambda_t}}\right)\right)\right],$$
$$0 \leq x < 1.$$

To illustrate the range of dynamics, Figure 4 shows the conditional distributions (conditional on no default) of the 5-year default probability and term hazard rate in one year for different parameter sets. The corresponding parameters are given in Table 2. In each case, the initial variance σ_0^2 and the function θ are calibrated to match default probabilities corresponding to an initial hazard rate of 3%.

Case (a) is a model with a deterministic time-change, which is just the Overbeck-Schmidt-model of Example 3.2. The initial 5-year default probability is approx. 0.14. It turns out that with a probability of roughly 1/2 the default probability in one year is below 0.05, and with a probability of roughly 1/5 it is above 0.2. Loosely speaking, with a high probability, the underlying entity will have either a very high or a very low credit quality in one year. This suggests a very volatile movement of the default probability process, which is explained as follows: In order to match the initial 5-year default probability, the credit quality process must

	(a)	(b)	(c)	(d)
		Parameters		
mean reversion a	3	3	1	1
default barrier b	-3	-3	-3	-2
jump intensity λ	0	2	1	0.0305
jump size distribution F		$\begin{bmatrix} 0.1\ (0.95) \\ 20\ \ (0.05) \end{bmatrix}$	$\begin{bmatrix} 0.1\ (0.95) \\ 10\ \ (0.05) \end{bmatrix}$	$\mathbf{1}_{\{[25000,\infty)\}}$
initial variance σ_0^2	3.16	4.59	3.25	10^{-6}
deterministic function θ	$\in [0.23, 1.32]$	$\in [-0.22, 0.04]$	$\in [-1.10, 0.37]$	0

Table 2. Parameters of dynamics examples. In each case, the jump process Z is a compound Poisson process with strictly positive jumps. The jump sizes are given by the first column and the corresponding probabilities in the second column of each matrix in the row of the jump size distribution F.

adopt a high default speed σ. The high volatility of the credit quality process then leads to a volatile default probability process.

Case (d) was chosen such that $\sigma_0^2 = 10^{-6}$ and $\theta \equiv 0$, so that the variance is very small until the first jump occurs. The jump size was chosen to be very large relative to the default barrier so that, heuristically, a single jump leads to default very quickly. Loosely speaking, case (d) can be considered an approximation of a reduced-form model with a deterministic and constant intensity: the credit quality process exhibits practically no movement, until the first jump occurs, which leads to default with a very high probability. This is also reflected in the jump intensity $\lambda = 3.05\%$, which is approximately the initial hazard rate, and in $\mathbf{P}(\tau \in (1,5)|X_1, \sigma_1^2) \approx 1 - \mathbf{e}^{-0.03 \cdot 4} = 0.11308$ conditional on no default until time 1.[1] These two cases illustrate that the parameters can be classified into parameters that govern the jump part of the variance process, namely the jump intensity and jump size distribution, and parameters that control the continuous behaviour of the process in the sense that level of the function θ determines the minimum volatility of the credit quality process at any time. By calibration to a term structure, a low level of jump activity leads to a higher minimum volatility and vice versa.

The characteristics of cases (b) and (c) are "in-between" cases (a) and (d): in both cases, the variance process exhibits jumps. However, the jump dynamics are moderate enough for the level of the variance process induced by θ and σ_0^2, both of which are obtained by calibration to the given term structure, to be significantly above zero. In other words, the variance processes of both cases feature jumps and a significant minimum "default speed".

The calibration error in the examples here is of the order of 10^{-6} basis points for cases (a)-(c) and 0.56 basis points for case (d).

6. Leveraged credit-linked note

As an example application of the model we value a leveraged credit-linked note using the pricing formula (3), with pricing done via Algorithm **1**. The note has a maturity of 5 years and a notional amount of €100. The leverage factor is $k = 5$, so that the payoff amount and time are linked to

[1]If the initial hazard rate is not constant, then a calibration where the variance moves purely by jumps cannot be attained. This is due to the fact that the jump intensity of the variance's compound Poisson process is constant, whereas a non-constant, deterministic hazard rate requires the jump intensity to be non-constant and deterministic. The former can be incorporated by specifying the jump process as an additive process.

	(a)		(b)		(c)		(d)	
$\tilde{k}$	5.000	(0.005)	4.674	(0.058)	4.865	(0.036)	1.943	(0.203)
s_{issuer} (bp)	0.60	(0.90)	58.60	(10.44)	24.31	(6.52)	550.19	(36.54)
s_{inv} (bp)	899.40	(0.90)	841.40	(10.44)	875.69	(6.52)	349.81	(36.54)
$\mathbb{E}(L_{\text{inv}})$ (€)	26.51	(1.57)	27.60	(0.73)	29.95	(10.66)	11.93	(0.75)
$\mathbf{P}(L_{\text{inv}})$	0.40	(0.024)	0.38	(0.01)	0.44	(0.015)	0.134	(0.008)
$\mathbf{P}(L_{\text{inv,tot}})$	0.002	(0.002)	0.067	(0.006)	0.030	(0.006)	0.134	(0.008)
$\mathbb{E}(E_{\text{inv}})$ (€)	26.88	(0.76)	27.28	(0.56)	25.20	(0.51)	14.44	(1.58)
$\mathbb{E}S$ (yrs)	0.88	(0.062)	1.46	(0.09)	0.93	(0.054)	2.41	(0.08)

Table 3. Valuation examples of leveraged credit-linked note. The cases (a)-(d) correspond to the models of Section 5.3.

the mark-to-market value of a CDS position with nominal €500 on CDS with a maturity of 5 years at inception. The trigger level is $K =$ €60. The initial CDS spread term structure is flat at 180 basis point, the recovery rate is 40%. The note is monitored weekly, i.e., at time points $t_1 < t_2 < \cdots < t_{260}$, with $\Delta t_i = 1/52$. Denote by V_t^k the mark-to-market value at time t of the CDS position. The note is unwound at $S = \inf\{t_i : V_{t_i}^k \leq -K, i = 1, \ldots, 260\}$. At S, the investor receives $\max(€100 + V_S^k, 0)$ and the issuer pays $\max(-V_S^k - €100, 0)$ (the gap option payoff). The spread of the gap option is $(k - \tilde{k})s(0,T)$, cf. Equation (4). The risk-free interest rate is constant at 5%.

For each dynamics example we generated 10 times 1000 simulations. From each batch of 1000 simulations, we computed the fair factor $\tilde{k}$, the spread of the gap option $s_{\text{issuer}} = (k - \tilde{k})s(0,T)$ and the spread on the note (for the investor) $s_{\text{inv}} = \tilde{k}s(0,T)$ for each of the four example models exhibiting different dynamics from Section 5.3. Additionally, denoting by L_{inv} the discounted loss to the investor, we computed the expected loss $\mathbb{E}(L_{\text{inv}})$, the probability that a loss occurs, the probability of a total loss $L_{\text{inv,tot}}$, the expected discounted earnings from the spread payments (excluding the default-free interest of the coupon payment), $\mathbb{E}(E_{\text{inv}})$, and the expected trigger time S conditional on a trigger event. The values obtained are given in Table 3; here, each table entry consists of the mean value taken over all runs and (in parentheses) the standard deviation with respect to the 10 simulation runs (the 10 simulation scenarios are simulations of the estimator, each of which is considered to be approximately normally distributed by the usual Central Limit Theorem).

Recall that in Section 2.4 we already determined the fair factor $\tilde{k}$ for some models via no-arbitrage arguments. Specifically, in the case where the mark-to-market value of a CDS evolves continuously, and when there is no jump-to-default risk, the fair factor is $\tilde{k} = k$, as there is no gap risk involved. This corresponds to case (a). Now consider the case where the mark-to-market value is constant and the note is exposed to default risk only by a jump-to-default event. Then $\tilde{k} = 1/(1-R)$ as the investor's payoff is equivalent to selling protection on $1/(1-R)$ CDS. This corresponds to case (d). Here, $\tilde{k}$ is slightly larger than $1/(1-R) = 1.67$ as there is still some, albeit small, volatility that drives the credit quality process, and consequently the underlying CDS's mark-to-market value is not constant. Also note that in this case the probabilities of a loss and of a total loss to the investor are approximately equal and correspond to the 1-year default probability $1 - \mathbf{e}^{-0.03 \cdot 5} = 0.139$. Finally, note that in this case, the expected trigger time conditional on a trigger event is roughly half of the note's maturity. In the other cases the expected trigger time is significantly earlier. There are apparently two reasons for this: first of all, the mark-to-market value of a CDS position vanishes as maturity is approached. Hence, the trigger event is less likely to occur for shorter remaining time-to-maturity. Secondly, in the absence of the possibility of large jumps in the variance process, calibration to the short end of the term structure may require rather high volatility in the credit quality process; this is a consequence of the fact that in the model short-term credit spreads vanish, whereas in reality they do not. This leads to high values of σ_0^2 and θ for short maturities. The resulting high volatility in the credit quality process affects the whole term structure, so that not only short-term, but also long-term credit spreads exhibit higher volatility in the short term.

Finally, observe that the efficiency of the simulation can be increased significantly as follows. Most of the mark-to-market value computations are used only to check whether the trigger level has been hit, and only in this case the actual mark-to-market value is needed. Compared to the number of total observations this is a rare event. Now, observe that the mark-to-market value is monotone in both X and σ^2. For each time step, and for a set of five values of the variance process σ_t^2 we computed the corresponding value of X_t for the mark-to-market value to be at the trigger level (actually, to be on the safe side, we computed X for a slightly smaller mark-to-market value). In each simulation, we then first checked against these computed valued whether the mark-to-market value needs to be computed at all. It turned out that in more than two thirds of evaluation steps the computation of credit spreads and mark-to-market value could be skipped.

7. Conclusion

We have introduced a first passage time model for credit spreads that is tractable and admits jumps in spreads. This allows for the efficient valuation of credit derivatives whose payoff depends explicitly on the level of credit spreads and, in particular, whose payoff is furthermore sensitive to the occurrence of jumps in credit spreads. An example of the latter type of product is the leveraged credit-linked note, which we describe in detail. We demonstrate that the model can be calibrated to a wide variety of term structures and dynamics, and we study the valuation of the leveraged credit-linked note under different setups.

Appendix A: Term hazard rate

We sometimes consider the *term hazard rate* defined by

$$\tilde{\lambda}(t,T) = -\frac{\ln(1-P(t,T))}{T-t}, \quad T > t, \quad P(t,T) < 1, \tag{A1}$$

as a proxy for the credit spread $s(t,T)/(1-R)$. The use of the term hazard rate is motivated by the fact that $\tilde{\lambda}(t,T)$ is a function of $P(t,T)$ instead of $(P(t,T))_{T\geq t}$ as is the case for the spread $s(t,T)$. That it may be considered a proxy for the credit spread is explained as follows: If the default time admits a (conditional) density, then the hazard rate at time t is the mapping $T \mapsto \lambda(t,T)$ defined by

$$\lambda(t,T) = -\frac{\mathrm{d}}{\mathrm{d}T}\ln(1-P(t,T)), \quad T \geq t.$$

It follows easily that $1 - P(t,T) = \mathbf{e}^{-\int_t^T \lambda(t,u)\,\mathrm{d}u}$, which, together with Equation (1) yields the relationship

$$\frac{s(t,T)}{1-R}\int_t^T \mathbf{e}^{-r(u-t)}(1-P(t,u))\,\mathrm{d}u = \int_t^T \lambda(t,u)\,\mathbf{e}^{-r(u-t)}(1-P(t,u))\,\mathrm{d}u.$$

An approximation of the right-hand side is $\tilde{\lambda}(t,T)\int_t^T \mathbf{e}^{-r(u-t)}(1-P(t,u))\,\mathrm{d}u$, with $\tilde{\lambda}(t,T) = \frac{\int_t^T \lambda(t,u)\,\mathrm{d}u}{T-t}$, which yields the well-known credit triangle $\frac{s(t,T)}{1-R} \approx \tilde{\lambda}(t,T)$.

Appendix B: Proofs

Proof: [Proposition 3.3] Let B be the DDS-Brownian motion of X, and recall that $B_{\Lambda_t} = X_t$, $t \geq 0$. By continuity of Λ and by properties of conditional expectation, $\mathbf{P}$–a.s., on $\{\tau > t\}$,

$$\begin{aligned}
\mathbf{P}(\tau \leq T|\mathcal{F}_t) &= \mathbf{P}\left(\min_{t<u\leq T} X_u \leq b\middle|\mathcal{F}_t\right) = \mathbf{P}\left(\min_{t<u\leq T} B_{\Lambda_u} \leq b\middle|\mathcal{F}_t\right)\\
&= \mathbf{P}\left(\min_{\Lambda_t<u\leq \Lambda_T} B_u \leq b\middle|\mathcal{F}_t\right) = \mathbf{P}\left(\min_{0<u\leq \Lambda_T-\Lambda_t} B_{\Lambda_t+u} \leq b\middle|\mathcal{F}_t\right)\\
&= \mathbf{P}\left(\min_{0<u\leq \Lambda_T-\Lambda_t} B_{\Lambda_t+u} - B_{\Lambda_t} \leq b - B_{\Lambda_t}\middle|\mathcal{F}_t\right)\\
&= \mathbb{E}\left(\mathbf{P}\left(\min_{0<u\leq \Lambda_T-\Lambda_t} B_{\Lambda_t+u} - B_{\Lambda_t} \leq b - B_{\Lambda_t}\middle|\mathcal{F}_t \vee \sigma(\Lambda_T)\right)\middle|\mathcal{F}_t\right). \qquad \text{(B1)}
\end{aligned}$$

The random time Λ_t is an $(\mathcal{F}_{\tau_t})_{t\geq 0}$-stopping time, and with $\mathcal{F}_{\tau_{\Lambda_t}} = \mathcal{F}_t$ it follows from Corollary C.4 that $(B_{\Lambda_t+u} - B_{\Lambda_t})_{u\geq 0}$ is a Brownian motion independent of $\mathcal{F}_t \vee \sigma(\Lambda_T) \subseteq \mathcal{F}_t \vee \mathcal{F}_\infty^\sigma$. On the other hand, the random variables $\Lambda_T - \Lambda_t$ and $b - B_{\Lambda_t}$ are $\mathcal{F}_t \vee \sigma(\Lambda_T)$-measurable. Hence, by the first passage time distribution of Brownian motion (see e.g. Section 2.6.A of Karatzas and Shreve (1998)), $\mathbf{P}$–a.s.,

$$\mathbf{P}\left(\min_{0<u\leq \Lambda_T-\Lambda_t} B_{\Lambda_t+u} - B_{\Lambda_t} \leq b - B_{\Lambda_t}\middle|\mathcal{F}_t \vee \sigma(\Lambda_T)\right) = 2\mathrm{N}\left(\frac{b - B_{\Lambda_t}}{\sqrt{\Lambda_T - \Lambda_t}}\right).$$

Inserting into Equation (B1) yields Equation (6).

For the first step of Equation (7), observe that on $\{\tau > t\}$ the events $\{t < \tau \leq T\}$ and $\{\min_{t<u\leq T} X_u \leq b\}$ are equal, and the latter is conditionally independent of $\mathcal{F}_t$ given (X_t, σ_t) by the Markov property. For the second step, taking into account that $\Lambda_T - \Lambda_t = \int_t^T \sigma_u^2\,\mathrm{d}u$, it follows that $2\mathrm{N}((b - X_t)/\sqrt{\Lambda_T - \Lambda_t})$ is a bounded random variable that is measurable with respect to $\sigma(X_u, u \geq t) \vee \sigma(\sigma_u, u \geq t)$. The claim now follows from Proposition 3.3 and the Markov property. □

Proof: [Proposition 3.4] That $\mathbf{P}(\int_0^t \sigma_s^2 \,\mathrm{d}s < \infty) = 1$, $t \geq 0$, is a consequence of

$$\int_t^T \sigma_u^2 \,\mathrm{d}u = \left(1 - \mathbf{e}^{-a(T-t)}\right) \frac{\sigma_t^2}{a} + \int_t^T \theta(u) \left(1 - \mathbf{e}^{-a(T-u)}\right) \mathrm{d}u + \frac{1}{a} \int_t^T \left(1 - \mathbf{e}^{-a(T-u)}\right) \mathrm{d}Z_u, \tag{B2}$$

and the fact that $Z_t < \infty$ $\mathbf{P}$–a.s. for any $t \geq 0$, since Z is a subordinator. Finally, $\lim_{t\to\infty} \int_0^t \sigma_s^2 \,\mathrm{d}s = \infty$, since for a subordinator Z, $\lim_{t\to\infty} Z_t = \infty$ $\mathbf{P}$–a.s. (see e.g. Section 3.1 of Bertoin (1998)). That (X, σ) is a Markov process follows from Theorem V.32 of Protter (2005). □

Proof: [Proposition 3.5] Abbreviate

$$h(t,T) = \int_t^T \theta(u) \left(1 - \mathbf{e}^{-a(t-u)}\right) \mathrm{d}u$$

$$L_{t,T} = \int_t^T \left(1 - \mathbf{e}^{-a(T-u)}\right) \mathrm{d}Z_u,$$

so that, by Equation (B2),

$$\Lambda_T - \Lambda_t = \left(1 - \mathbf{e}^{-a(T-t)}\right) \frac{\sigma_t^2}{a} + h(t,T) + \frac{L_{t,T}}{a}.$$

On $\{\tau > t\}$ and by the Markov property of (X, σ), the conditional default probability at t until T is given by Equation (7), and by the independence of $L_{t,T}$ and (X, σ), a version of this conditional probability is given by $g_{t,T}(X_t, \sigma_t)$ with

$$g_{t,T}(x,y) := \mathbb{E}\left(2\mathrm{N}\left(\frac{b-x}{\sqrt{\left(1 - \mathbf{e}^{-a(T-s)}\right) y^2/a + h(t,T) + L_{t,T}/a}}\right)\right). \tag{B3}$$

To derive the claim of the Proposition we require the following:

(i) $L_{t-,T} = L_{t,T}$ $\mathbf{P}$–a.s., for any $t \leq T$,
(ii) for any sequence $(t_n, x_n, y_n) \to (t, x, y)$,

$$g_{t_n,T}(x_n, y_n) \to g_{t,T}(x,y), \quad \mathbf{P}\text{–a.s.}, \tag{B4}$$

(iii) for $(b - x) < 0$, $g_{t,T}(x,y)$ is strictly increasing in y.

Property (i) is well-known, see e.g. (Sato 1999, p. 6). For (ii) observe that

$$\frac{b - x_n}{\sqrt{\left(1 - \mathbf{e}^{-a(T-t_n)}\right) y_n^2/a + h(t_n,T) + L_{t_n,T}/a}} \to \frac{b-x}{\sqrt{\left(1 - \mathbf{e}^{-a(T-t)}\right) y^2/a + h(t,T) + L_{t,T}/a}}, \quad \mathbf{P}\text{–a.s.}, \quad \text{as } n \to \infty,$$

by (i) and since all the terms in the sum of the denominator converge and the limit of the denominator is greater 0, for $t < T$. Equation (B4) is obtained by continuity of the Normal distribution and Dominated Convergence.

For (iii) observe that the denominator in Equation (B3) is strictly increasing in y and that for $(b-x) < 0$, $t \mapsto \mathrm{N}((b-x)/\sqrt{t})$ is strictly increasing.

Fix $g_{t,T}(X_t, \sigma_t)$ as the version of the conditional default probability $P(t,T)$ on $\{\tau > t\}$. Then, taking into account that X is continuous $\mathbf{P}$–a.s., and that on $\{\tau > t\}$ we have $(b - X_t) < 0$, we obtain $\mathbf{P}$–a.s. for any sequence $t_n \uparrow t$,

$$P(t-,T) = \lim_{t_n \uparrow T} g_{t_n,T}(X_{t_n}, \sigma_{t_n}) = g_{t,T}(X_t, \sigma_{t-}) \begin{Bmatrix} = g_{t,T}(X_t,\sigma_t), & \text{if } \Delta\sigma_t = 0 \\ < g_{t,T}(X_t,\sigma_t), & \text{if } \Delta\sigma_t > 0 \end{Bmatrix} = P(t,T).$$

□

Proof: [Proposition 3.6] The claim is derived using the relationship between the CDS spread and conditional default probabilities, Equation (1). Consider first the integral of the numerator of Equation (1). For any sequence $t_n \to t$, as $n \to \infty$, we have

$$\lim_{n\to\infty} \int_{t_n}^{T} r\,\mathbf{e}^{-r(u-t_n)}\,P(t_n,u)\,\mathrm{d}u = \lim_{n\to\infty} \int_{t}^{T} r\,\mathbf{e}^{-r(u-t_n)}\,P(t_n,u)\,\mathrm{d}u.$$

Then, for the numerator of Equation (1), it follows by Dominated Convergence, for any sequence $t_n \uparrow t$ as $n \to \infty$,

$$\lim_{n\to\infty} \left[\mathbf{e}^{-r(T-t_n)} P(t_n,T) + \int_t^T r\,\mathbf{e}^{r(u-t_n)}\,P(t_n,u)\,\mathrm{d}u \right] = \mathbf{e}^{-r(T-t)} P(t-,T) + \int_t^T r\,\mathbf{e}^{-r(u-t)}\,P(t-,u)\,\mathrm{d}u.$$

Similarly, we obtain for the denominator,

$$\lim_{n\to\infty} \int_{t_n}^{T} \mathbf{e}^{-r(u-t_n)}\,(1 - P(t_n,u))\,\mathrm{d}u = \int_t^T \mathbf{e}^{-r(u-t)}\,(1 - P(t-,u))\,\mathrm{d}u.$$

It follows that

$$\begin{aligned} \frac{\Delta s(t,T)}{1-R} &= \frac{\mathbf{e}^{-r(T-t)} P(t,T) + \int_t^T r\,\mathbf{e}^{-r(u-t)}\,P(t,u)\,\mathrm{d}u}{\int_t^T \mathbf{e}^{-r(u-t)}\,(1-P(t,u))\,\mathrm{d}u} \\ &\quad - \frac{\mathbf{e}^{-r(T-t)} P(t-,T) + \int_t^T r\,\mathbf{e}^{-r(u-t)}\,P(t-,u)\,\mathrm{d}u}{\int_t^T \mathbf{e}^{-r(u-t)}\,(1-P(t-,u))\,\mathrm{d}u} \\ &\begin{cases} = 0, & \text{if } \Delta P(t,u) = 0, \quad t < u \le T \\ > 0, & \text{if } \Delta P(t,u) > 0, \quad t < u \le T. \end{cases} \end{aligned} \tag{B5}$$

In particular, this establishes that $\Delta s(t,T) > 0 \Longrightarrow (\Delta P(t,u) > 0,$ for some $u \in (t,T])$. For the converse statement observe that $\Delta P(t,u) > 0$ for some $u \in (t,T]$ if and only if $\Delta P(t,u) > 0$ for all $u \in (t,T]$ by the previous Proposition and by Equation (B5). □

Proof: [Proposition 5.2] We have $\{P(t,T) \leq x\} = \big\{\mathbf{P}(\tau \in (t,T] | X_t, \sigma_t^2) \leq x, \tau > t\big\}$, which establishes

$$\mathbf{P}(P(t,T) \leq x | \tau > t) = \frac{\mathbf{P}(P(t,T) \leq x)}{\mathbf{P}(\tau > t)}, \quad x < 1.$$

By the independence of the DDS-Brownian motion B and the pair σ_t^2, Λ_t, we obtain on $\{\tau > t\}$,

$$\begin{aligned}
\mathbf{P}\left(P(t,T) \leq x | \Lambda_t, \sigma_t^2\right) &= \mathbf{P}\left(f_{\sigma_t^2}(X_t) \leq x, \min_{0<s\leq t} X_s > b \middle| \Lambda_t, \sigma_t^2\right) \\
&= \mathbf{P}\left(f_{\sigma_t^2}(B_{\Lambda_t}) \leq x, \min_{0<s\leq \Lambda_t} B_s > b \middle| \Lambda_t, \sigma_t^2\right) \\
&= \mathbf{P}\left(B_{\Lambda_t} \geq f_{\sigma_t^2}^{(-1)}(x), \min_{0<s\leq \Lambda_t} B_s > b \middle| \Lambda_t, \sigma_t^2\right) \\
&= \mathbf{P}\left(B_{\Lambda_t} \geq f_{\sigma_t^2}^{(-1)}(x) \middle| \Lambda_t, \sigma_t^2\right) - \mathbf{P}\left(B_{\Lambda_t} \geq f_{\sigma_t^2}^{(-1)}(x), \min_{0<s\leq \Lambda_t} B_s \leq b \middle| \Lambda_t, \sigma_t^2\right) \\
&= \mathrm{N}\left(\frac{-f_{\sigma_t^2}^{(-1)}(x)}{\sqrt{\Lambda_t}}\right) - \mathrm{N}\left(\frac{2b - f_{\sigma_t^2}^{(-1)}(x)}{\sqrt{\Lambda_t}}\right),
\end{aligned}$$

using the joint distribution of a Brownian motion and its running minimum (see e.g. Sections 2.6.A and 2.8.A of Karatzas and Shreve (1998)). The claim now follows by taking expectation. □

Proof: [Lemma 4.1] Conditioning under Y yields

$$F(x) = \mathbf{P}\left(\left(1 - \mathbf{e}^{-a(T-S)}\right) Y \leq x\right) = \mathbb{E}\left(\mathbf{P}\left(\left(1 - \mathbf{e}^{-a(T-S)}\right) Y \leq x \middle| Y\right)\right). \tag{B6}$$

Define $g_x(y) := \mathbf{P}\left(\left(1 - \mathbf{e}^{-a(T-S)}\right) y \leq x\right)$, $y > 0$. By the independence of S and Y, the conditional probability of Equation (B6) is given by $g_x(Y)$. Since $S \in [0,T]$,

$$g_x(y) = \begin{cases} 0, & x \leq 0, \\ 1, & x \geq (1 - \mathbf{e}^{-aT})y, \\ -\frac{\ln(1-x/y)}{aT}, & x \in (0, (1 - \mathbf{e}^{-aT})y), \end{cases}$$

and the claim follows by inserting into Equation (B6). □

Appendix C: Ocone martingales

The famous Theorem of Dambis, Dubins-Schwarz (DDS-Theorem) (see e.g. (Karatzas and Shreve 1998, Chapter 3.4.B)) states that a continuous local martingale has a representation as a time-changed Brownian motion. *Ocone martingales* are those continuous local martingales whose DDS-Brownian motion and associated time-change are independent, see Ocone (1993), Dubins *et al.* (1993), Vostrikova and Yor (2000). We study the relationship between stochastic integrals (with respect to Brownian motion) and Ocone martingales. The result is applied in Section 3.2 to derive conditional default probabilities in the credit quality process model.

Assume given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ that satisfies the usual hypotheses. For a continuous local martingale M, let us call the Brownian motion in its representation as a time-changed Brownian motion the *DDS-Brownian motion of* M, and let us denote denote the quadratic variation of M by $[M, M]$.

Definition C.1: A continuous local martingale $M = (M_t)_{t\geq 0}$, $M_0 = 0$, is an *Ocone martingale*, if its DDS-Brownian motion is independent of $[M, M]$.

We are interested in identifying Ocone martingales that are defined as stochastic integrals with respect to a Brownian motion. Vostrikova and Yor (2000) provide a characterisation for adapted, continuous integrands. We shall allow adapted càdlàg processes as integrands.

Proposition C.2: *Let $W = (W_t)_{t\geq 0}$ be an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion and let $\sigma = (\sigma_t)_{t\geq 0}$ be an $(\mathcal{F}_t)_{t\geq 0}$-adapted, strictly positive càdlàg process such that $\mathbf{P}(\int_0^t \sigma_s^2 \,\mathrm{d}s < \infty) = 1$, $t \geq 0$, and $\mathbf{P}(\lim_{t\to\infty} \int_0^t \sigma_s^2 \,\mathrm{d}s = \infty) = 1$. Let $M = (M_t)_{t\geq 0}$ be the continuous local martingale given by*

$$M_t = \int_0^t \sigma_s \,\mathrm{d}W_s, \quad t \geq 0.$$

Then, W and σ are independent if and only if M is an Ocone martingale.

Observe that $[M, M] = \int_0^\cdot \sigma_s^2 \,\mathrm{d}s$. For the proof, we need the following Lemma.

Lemma C.3: *Let W, σ, M be as in Proposition C.2. Then, $\sigma([M, M]_t, 0 \leq t < \infty) = \sigma(\sigma_t^2, 0 \leq t < \infty)$.*

Proof: Clearly, $\sigma([M, M]_t, 0 \leq t < \infty) \subseteq \sigma(\sigma_t^2, 0 \leq t < \infty)$. For the converse statement, observe first that $[M, M]$ is differentiable a.e. (with respect to Lebesgue measure), and for every such t, $[M, M]'_t = \sigma_t^2$. Conversely, at any point t where σ^2 is continuous, $[M, M]$ is differentiable, for suppose that σ^2 is continuous at t, but

$$\lim_{h\downarrow 0} \frac{[M, M]_{t+h} - [M, M]_t}{h} \neq \lim_{h\uparrow 0} \frac{[M, M]_{t+h} - [M, M]_t}{h};$$

this is a contradiction to $[M, M]' = \sigma^2$ a.e. and the continuity of σ^2 at t. Now, let t be a point where $[M, M]$ is not differentiable; i.e., σ^2 has a jump at t. From the right-continuity of σ^2, it follows that there exists $h > 0$ such that $[M, M]$ is differentiable on $(t, t+h)$. By the Mean Value Theorem, there exists $s_h \in (t, t + h)$ such that

$$\frac{[M, M]_{t+h} - [M, M]_t}{h} = \sigma_{s_h}^2.$$

Moreover, by the right-continuity of σ^2, taking the right limit,

$$\lim_{h\downarrow 0} \frac{[M, M]_{t+h} - [M, M]_t}{h} = \lim_{h\downarrow 0} \sigma_{s_h}^2 = \sigma_t^2,$$

and $\sigma(\sigma_t^2, 0 \leq t < \infty) \subseteq \sigma([M, M]_t, 0 \leq t < \infty)$ follows (see e.g. Lemma 1.13 of Kallenberg (2001)). □

Proof: [Proposition C.2] Define the family of stopping times $(\tau_t)_{t\geq 0}$ by

$$\tau_t = \inf\left\{s \geq 0 : \int_0^s \sigma_u^2 \, \mathrm{d}u > t\right\}, \quad t \geq 0.$$

Then $B = (B_t)_{t\geq 0}$, given by $B_t = M_{\tau_t} = \int_0^{\tau_t} \sigma_s \, \mathrm{d}W_s$, $t \geq 0$, is the DDS-Brownian motion of M. It is an $(\mathcal{F}_{\tau_t})_{t\geq 0}$-Brownian motion.

The "only if" part: Let C be the space of real-valued continuous functions on $\mathbb{R}_+$ and let D be the space of real-valued càdlàg functions on $\mathbb{R}_+$. Denote by $(C, \mathcal{B}(C))$, resp. $(D, \mathcal{B}(D))$, the measurable space of real-valued continuous, resp. càdlàg , functions on $\mathbb{R}_+$ endowed with the σ-algebra generated by the finite-dimensional cylinder sets of C, resp. D.[1] For every $\Gamma \in \mathcal{B}(C)$ and $\Delta \in \mathcal{B}(D)$ we show that

$$\mathbf{P}(B \in \Gamma, \sigma \in \Delta) = \mathbf{P}(B \in \Gamma)\, \mathbf{P}(\sigma \in \Delta). \tag{C1}$$

That B is independent of $[M, M]$ and hence an Ocone martingale then follows by Lemma C.3.

It is straightforward that Equation (C1) holds for sets Δ with $\mathbf{P}(\sigma \in \Delta) \in \{0, 1\}$. Choose Δ such that $\mathbf{P}(\sigma \in \Delta) \in (0, 1)$, and denote by $\mathscr{D}$ the σ-algebra generated by $\{\sigma \in \Delta\}$. By properties of conditional expectation,

$$\mathbf{P}(B \in \Gamma, \sigma \in \Delta) = \mathbb{E}(\mathbf{1}_{\{\sigma\in\Delta\}}\mathbf{P}(B \in \Gamma|\mathscr{D})). \tag{C2}$$

Writing $D_1 = \{\sigma \in \Delta\}$ and $D_2 = \{\sigma \notin \Delta\}$, it is easy to check that a version of the conditional probability of $A \in \mathcal{F}$ with respect to $\mathscr{D}$ is given by

$$\mathbf{P}(A|\mathscr{D})(\omega) = \sum_{i=1,2} \mathbf{P}(A \cap D_i)/\mathbf{P}(D_i)\mathbf{1}_{\{D_i\}}(\omega), \quad \omega \in \Omega.$$

Fix this version of the conditional probability. For every $\omega \in \Omega$, $\mathbf{P}(\cdot|\mathscr{D})(\omega)$ is a probability measure (and thus it is a variant of the regular conditional probability with respect to $\mathscr{D}$. Moreover, $\mathbf{P}(\cdot|\mathscr{D})(\omega) \ll \mathbf{P}$, i.e., $\mathbf{P}(\cdot|\mathscr{D})(\omega)$ is absolutely continuous with respect to $\mathbf{P}$. It follows, e.g. by Theorem 14 of (Protter 2005, Section II.5), that $\int_0^\cdot \sigma_s \, \mathrm{d}W_s$ computed under the law $\mathbf{P}(\cdot|\mathscr{D})(\omega)$ and M are $\mathbf{P}(\cdot|\mathscr{D})(\omega)$-indistinguishable.

By independence of W and $\mathscr{D}$ it follows that W is a Brownian motion under $\mathbf{P}(\cdot|\mathscr{D})(\omega)$, and hence M is a continuous local martingale under $\mathbf{P}(\cdot|\mathscr{D})(\omega)$. The quadratic variation, as a limit in probability, is invariant to absolutely continuous changes in measure. Hence, by the Lévy-characterisation of Brownian motion, B is an $(\mathcal{F}_{\tau_t})$-Brownian motion under $\mathbf{P}(\cdot|\mathscr{D})(\omega)$, in other words $\mathbf{P}(B \in \cdot|\mathscr{D}) = \mathbf{P}(B \in \cdot)$ is the Wiener measure on $(C, \mathcal{B}(C))$. Finally, inserting into Equation (C2) yields

$$\mathbb{E}(\mathbf{1}_{\{\sigma\in\Delta\}}\, \mathbf{P}(B \in \Gamma|\mathscr{D})) = \mathbb{E}(\mathbf{1}_{\{\sigma\in\Delta\}}\, \mathbf{P}(B \in \Gamma)) = \mathbf{P}(B \in \Gamma)\, \mathbf{P}(\sigma \in \Delta). \tag{C3}$$

The "if" part: Now suppose that M is an Ocone martingale, i.e., B and $[M, M]$ are independent. We have

$$W_t = \int_0^t \frac{1}{\sigma_s} \, \mathrm{d}M_s = \int_0^{[M,M]_t} \frac{1}{\sigma_{\tau_s}} \, \mathrm{d}B_s, \quad \mathbf{P}\text{–a.s.},$$

[1] Under a suitable metric on C, resp. D, the σ-algebra $\mathcal{B}(C)$, resp. $\mathcal{B}(D)$, corresponds to the σ-algebra generated by the open sets (with respect to the metric) of C, resp. D, see e.g. (Shiryaev 1996, Section II.§2) or (Karatzas and Shreve 1998, Sections 2.4 and 6.2).

where the last part follows from (Karatzas and Shreve 1998, Proposition 3.4.8). Now it can be shown that, for $\Gamma, \Delta \in \mathcal{B}(C)$),

$$\mathbf{P}(W \in \Gamma, [M,M] \in \Delta) = \mathbf{P}(W \in \Gamma)\, \mathbf{P}([M,M] \in \Delta)$$

using the same technique as in the "only if" part of the proof. That W and σ are independent then follows from Lemma C.3. □

In Proposition C.2 we started out with a continuous local martingale defined as a stochastic integral with respect to a Brownian motion. We may also consider the converse, where we define a continuous local martingale M as a Brownian motion and an independent time-change. Then, a sufficient condition for M to have a representation as a stochastic integral with respect to a Brownian motion is that $[M,M]$ be an absolutely continuous function of t for $\mathbf{P}$-almost every $\omega \in \Omega$, cf. (Karatzas and Shreve 1998, Theorem 3.4.2).

We conclude this section by establishing some properties related to Proposition C.2 that are used in Section 3.2. For a stochastic process X, we denote by $(\mathcal{F}_t^X)_{t\geq 0}$ the filtration generated by X.

Corollary C.4: *Let $M = \int_0^\cdot \sigma_s \,\mathrm{d}W_s$ be as in Proposition C.2, and let B the DDS-Brownian motion of M. Define the family of stopping times $(\tau_t)_{t\geq 0}$ by $\tau_t = \inf\left\{s \geq 0 : \int_0^s \sigma_u^2 \,\mathrm{d}u > t\right\}$, $t \geq 0$. Furthermore, let S be an $\mathbf{P}$–a.s. finite $(\mathcal{F}_{\tau_t})$-stopping time and define $\tilde{B} = (\tilde{B}_u)_{u\geq 0}$, with $\tilde{B}_u := B_{S+u} - B_S$. If M is an Ocone martingale, then*

(i) $\tilde{B}$ is an $(\mathcal{F}_t^{\tilde{B}})_{t\geq 0}$-Brownian motion independent of σ;
(ii) $(\mathcal{F}_t^{\tilde{B}})_{t\geq 0}$ and $\mathcal{F}_{\tau_S}$ are conditionally independent given $\mathcal{F}_\infty^\sigma$, the σ-algebra generated by σ;
(iii) $\tilde{B}$ is independent of $\mathcal{F}_{\tau_S} \vee \mathcal{F}_\infty^\sigma$, the smallest σ-algebra containing $\mathcal{F}_{\tau_S}$ and $\mathcal{F}_\infty^\sigma$.

Proof:

(i) By the properties of Brownian motion, $\tilde{B}$ is a Brownian motion independent of $\mathcal{F}_{\tau_S}$, cf. (Karatzas and Shreve 1998, Theorem 2.6.16). In the notation of the previous proof, let $\Delta \in \mathcal{B}(D)$ and let $\mathscr{D}$ be the σ-algebra generated by $\{\sigma \in \Delta\}$. Since B is a Brownian motion under $\mathbf{P}(\cdot|\mathscr{D})(\omega)$, for $\mathbf{P}$-almost all $\omega \in \Omega$, so is $\tilde{B}$, and the first claim follows in analogy to Equation (C3).

(ii) The second claim is a generalisation of the first claim. Since B is independent of σ, we have $\mathbf{P}(B \in \Gamma|\mathcal{F}_\infty^\sigma) = \mathbf{P}(B \in \Gamma)$ $\mathbf{P}$–a.s., $\Gamma \in \mathcal{B}(C)$. Moreover, there exists a version $\mathbf{Q}$ of $\mathbf{P}(B \in \cdot|\mathcal{F}_\infty^\sigma)$ that is a regular conditional distribution of B with respect to $\mathcal{F}_\infty^\sigma$. It follows that B is a Brownian motion under $\mathbf{Q}$ and hence $\tilde{B}$ is a Brownian motion independent of $\mathcal{F}_{\tau_S}$ under $\mathbf{Q}$, which is the required result.

(iii) Since $\mathcal{F}_\infty^{\tilde{B}}$ and $\mathcal{F}_{\tau_S}$ are conditionally independent given $\mathcal{F}_\infty^\sigma$, $\mathbf{P}$–a.s. (see Proposition 6.6 of Kallenberg (2001) for the first statement),

$$\mathbf{P}(\tilde{B} \in \Gamma|\mathcal{F}_\infty^\sigma \vee \mathcal{F}_{\tau_S}) = \mathbf{P}(\tilde{B} \in \Gamma|\mathcal{F}_\infty^\sigma) = \mathbf{P}(\tilde{B} \in \Gamma), \quad \Gamma \in \mathcal{B}(C).$$

□

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