Static hedging under maturity mismatch

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Abstract

Can shorter maturity European options be statically hedged with longer maturity plain vanilla options? This problem appears for example when analyzing options on forwards in relation to liquid options on the spot underlying. Under mild assumptions on the underlying security price process and on the option's payoff function we show that approximate static hedges exist und we provide a recipe for constructing them. Examples illustrate the power of the hedge and its sensitivity to modelling assumptions. The results can be extended to formulating semi-static hedging strategies for discretely monitored path-dependent contingent claims.

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1 Introduction

We consider the problem of statically hedging the payoff $p(S_t)$ of a European option maturing at time t written on some asset S_t. It is well-known that any sufficiently regular contingent claim $p(S_t)$ can be hedged by a static portfolio of standard options (that is, European calls or puts) with maturity t, if these are assumed to be liquid for all strikes, see e.g. (Carr and Madan, 1998; Albrecher and Mayer, 2010). However, the standard options market in general cannot be expected to match the payoff's expiry and might only be liquid for some maturity $T > t$. This is the situation under which we investigate the static hedging problem. Interestingly, even when presuming standard options expiring at $T > t$ to be liquid for all strike levels, it is not clear which, if any, contingent claims $p(S_t)$ can be hedged by a portfolio of these options.. This is because, in contrast to the case $T = t$, no such thing as Taylor's formula can be employed to derive the hedge.

Static hedging itself is motivated by practical applications. If a market provides liquidly traded options that themselves qualify as hedging instruments, then static hedging is a viable alternative to dynamic hedging, (Derman et al., 1995; Carr and Bowie, 1994). First, static replication portfolios are often less sensitive to model assumptions than dynamic replicating strategies. Second, for derivatives with high gamma delta-hedging can yield unreasonably high P&L due to both discretization error and potentially substantial transaction costs arising from rebalancing. However, one often finds a maturity mismatch between the OTC option to be hedged and the liquid options suitable for static hedging. Here, we consider the case of hedging illiquid short-dated options given a liquid options market for longer maturities. A prominent

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example for this situation pops up in the interest rate market, when hedging and pricing highly illiquid options on forward starting swaps using standard swaptions. More generally, considering a forward contract maturing at time T, one can relate time-t-maturity options on that forward contract to standard (spot) options maturing at $T > t$. The converse case where the liquid options have a shorter maturity is treated in Carr and Wu (2013).

On the technical level, we first consider the problem of statically hedging a European payoff with longer-dated European call options of arbitrary strikes. Throughout we assume that the underlying price process is an exponential additive process. Using Fourier inversion techniques, we show that for a general payoff function p, there exists a sequence of hedging strategies that converges almost surely to the desired payoff. Furthermore, if the payoff fulfills an integrability condition, then there exists a sequence of hedging strategies that converges to $p(S_t)$ in L^1. In practice, L^1 convergence is often considered more useful, as it bounds the magnitude of the expected hedging error.

Although these results state the existence of hedging strategies in an asymptotic sense, they do not devise the construction of an explicit strategy. In fact, for many typical payoffs, including standard payoffs such as call options, the hedging strategy derived from Fourier inversion is not in L^1 and straightforward approximations yield highly oscillatory hedging strategies involving an unreasonably high number of traded hedging instruments. To address these issues, we construct feasible hedging strategies using Tikhonov regularization and show that – under some further moderate assumptions on the respective payoff functions – these strategies converge in L^1 to the desired payoffs. The so-constructed hedges are demonstrated in a number of examples.

All strategies mentioned above assume that a continuum of hedging instruments is tradable. For applications in practice this of course is not a feasible working assumption and approximate static hedges only employing finitely many strikes are therefore of enormous practical importance. In a second step, we thus consider the static hedging problem in a discrete setting and compare different discretization techniques to construct explicit hedging strategies.

In contrast to most static hedging strategies available in the literature, the hedging strategy and error in our setup are subject to model (mis-)specifications. This is because the hedging instruments are longer dated than the target payoff, and a model assumption is required for pricing the hedging instruments at the payoff's maturity. Throughout, we compare the hedging strategies in the Black-Scholes model and the CGMY model, and we find that – as long as the maturity mismatch is not too large – the error introduced by the model assumption is sufficiently small to be negligible. Finally, we compare the error from static hedging, which is introduced by assuming a finite number of hedging instruments, to the error from dynamic hedging, which is introduced from discrete rebalancing and possibly from model incompleteness. Our results indicate that dynamic hedging outperforms static hedging in a Black-Scholes model, whereas static hedging dominates dynamic delta-hedging in the CGMY model, where the asset price process is jump-driven. In the case of the Heston model, that is, an incomplete diffusion model, the dynamic and static hedging produce P&L of similar magnitude.

Although the static hedging method derived is restricted to European payoffs, it can be easily extended to construct semi-static hedges for discretely monitored path-dependent payoffs. Essentially, the semi-static hedging strategy is made of static hedges between each of the monitoring time points, and where the "payoff" to be hedged until the ensuing time point corresponds to the option value as a function of the underlying asset price.

To put this paper into perspective we would like to take the opportunity to mention some work in the field of static hedging. However, since the literature is vast and we cannot do justice to the numerous original research papers, we confine ourselves to mention the closest and most important one for the specific topic here.

For a sufficiently smooth payout $p(S_t)$, a static hedge with plain vanilla options maturing at time t can be built as a straightforward consequence of the Taylor expansion of p, a fact that traces back to the seminal paper by Breeden and Litzenberger (1978) and was further developed by e.g. Green and Jarrow (1987), Nachman (1988) and Carr and Madan (1998). This hedge is model-independent but employs vanilla options at a continuum of strikes. For exotic options the idea of static hedging appeared first in the mid 1990s in the seminal papers Carr and Bowie (1994) and Derman et al. (1995). Subsequently, the idea has been extended under a variety of modelling assumptions and for other exotic products, with barrier options attracting most attention (e.g. Carr et al., 1998). Studies comparing the performance of static and dynamic hedging strategies are conducted e.g. in Tompkins (2002), Nalholm and Poulsen (2006) and Engelmann et al. (2006). A survey of semi-static hedging strategies with the further references is given in Albrecher and Mayer (2010).

Probably closest to our paper is Carr and Wu (2013), who study semi-static hedging under the setup of shorter-dated hedging instruments. In their setup, the required position in each of the shorter-term options depends on the gamma of the target payoff at expiry of the short-term option. Similar to our results, Carr and Wu (2013) find that static hedging outperforms delta hedging in incomplete models with jumps. Because in the setup of Carr and Wu (2013), the payoff is longer-dated than the hedging instruments, the problem of high oscillations mentioned above does not arise and consequently a perfect static hedge – always under the assumption of a continuum of call options available for hedging – exists under fairly general conditions. Thus, hedging with longer-dated options introduces additional imperfections that do not arise when hedging with shorter-dated options.

The paper is organized as follows. In Section 2 we introduce the model setup for the underlying price process and we provide technical prerequisites on the Fourier transform associated with the driving process of our model. Under mild conditions on the payoff function we show existence of approximating static hedging strategies in Section 3. The actual construction of approximating hedge portfolios is addressed in Section 4. Examples in the Black & Scholes and the CGMY modelling framework illustrate the construction. Section 5 deals with hedges for which only a finite number of given plain vanilla hedging instruments is available. In-line with Carr and Wu (2013) we consider a Gaussian quadrature discretisation of the continuous hedging strategy, and we further consider a state-space discretisation, where the optimal hedge is obtained as the solution of a constrained linear problem. Section 6 provides examples for various payoffs and compares the hedging performance assuming both Black-Scholes type and CGMY models. Furthermore, the hedging errors from static and dynamic hedging in Black-Scholes, CGMY and Heston-models are compared, which provides insight into the usefulness of static hedging compared to dynamic hedging as an alternative. Finally, we conclude in Section 7.

2 Setup and preliminaries

Our model setup is as follows: The asset price process $(S_t)_{t\geq 0}$ is defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$. We assume that $\mathbb{Q}$ is a risk-neutral measure and $(S_t)_{t\geq 0}$ is modelled by some exponential additive process, i.e.,

$$S_t = S_0 \mathrm{e}^{(r-q)t+X_t}, \tag{1}$$

where S_0 is the starting price of the asset, r is the riskless interest rate, q is the dividend yield and $X_t = Y_{A(t)}$ with $(Y_t)_{t\geq 0}$ being some Lévy process with characteristic triplet (μ, σ^2, ν), such that $\exp\{Y_t\}$ is a martingale, and $A(t) = \int_0^t a(s)\ ds$ for some deterministic, positive and bounded function a.

Note that we need to specify the risk-neutral measure, as the hedging portfolio must be evaluated at the maturity of the contingent claim being hedged. Nevertheless, as the risk-neutral measure is equivalent to the objective one $\mathbb{P}$ the almost sure convergence put forth in Theorem 1 translates exactly to $\mathbb{P}$. Hence, also what follows can easily be translated to the objective measure, which is the classic one for evaluating hedging problems.

This setup implies that the returns of the asset price are independent, which is crucial for the results of the paper. This class of models includes models based on Lévy processes, such as jump-diffusion models, (Merton, 1976; Kou, 2002), the variance gamma model, (Madan and Seneta, 1990; Madan et al., 1998), the normal inverse Gaussian model, (Barndorff-Nielsen, 1998, 1997), generalized hyperbolic models, (Barndorff-Nielsen, 1977; Eberlein and Raible, 2001; Eberlein, 2001) and the CGMY model, (Carr et al., 2002; Boyarchenko and Levendorskii, 2002). It does not include for example local volatility models, (Dupire, 1994), and stochastic volatility models such as the Heston model, (Heston, 1993).

The analysis can be carried out equally well when the additive process $(X_t)_{t\geq 0}$ is replaced by a general semimartingale with independent increments, since then the characteristic function of the process is also deterministic, cf. (Jacod and Shiryaev, 2003), and of similar kind as for the additive processes considered here. However, for the ease of exposition we limit ourselves to additive processes and leave the straightforward extension to the slightly more general processes to the reader.

For the ease of exposition we assume $r = q = 0$ and $S_0 = 1$ throughout the paper. The two maturities t, T with $T > t$ are fixed.

We find it convenient to express the payoffs and prices of European options in terms of the driving additive process X_t. For example, we denote the time-t price of a European call with strike K and maturity T by

$$\mathrm{C}(X_t, K) = \mathbb{E}\left[\left(\mathrm{e}^{X_T} - K\right)^+ \middle| \mathcal{F}_t\right]. \tag{2}$$

Solving the hedging problem via Fourier techniques requires some further technical assumptions. In order to satisfy square-integrability conditions later on, we make the following standard assumption, cf. (Carr and Madan, 1999).

Assumption 1. *There exists an $\alpha > 0$, such that*

$$\mathbb{E}\left[e^{(1+\alpha)X_T}\right] < \infty. \tag{3}$$

Remark 1. Note that $\mathbb{E}\left[\mathrm{e}^{(1+\alpha)X_T}\right] < \infty$ implies $\mathbb{E}\left[\mathrm{e}^{(1+\alpha)(X_T - X_s)}\right] < \infty$ and $\mathbb{E}\left[\mathrm{e}^{(1+\alpha)X_s}\right] < \infty$, for all $0 \leq s \leq T$.

We define the function $\gamma : \mathbb{R} \to \mathbb{R}$ by

$$\gamma(k) = \mathrm{e}^{-\alpha k}\mathbb{E}\left[(\mathrm{e}^{X_T - X_t} - \mathrm{e}^{-k})^+\right] = \mathrm{e}^{-\alpha k}\,\mathrm{C}(X_t, \mathrm{e}^{-k+X_t})\mathrm{e}^{-X_t}, \tag{4}$$

with $\alpha > 0$ from (3). Note that writing $\gamma(k)$ is justified, because of the independent increments of X, the right hand side does actually not depend on X_t. The function γ will play a crucial role in the Fourier approach later.

Proposition 1. *We have $\gamma \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.*

Proof. First, we show that $\gamma \in L^\infty(\mathbb{R})$:

$$\begin{aligned}\gamma(k) = \mathrm{e}^{-\alpha k}\mathbb{E}[(\mathrm{e}^{(X_T - X_t)} - \mathrm{e}^{-k})^+] &= \mathbb{E}[(\mathrm{e}^{(X_T - X_t) - \alpha k} - \mathrm{e}^{-(1+\alpha)k})^+] \\ &\leq \mathbb{E}[(\mathrm{e}^{(1+\alpha)(X_T - X_t)} - \mathrm{e}^{-(1+\alpha)k})^+] \\ &\leq \mathbb{E}[\mathrm{e}^{(1+\alpha)(X_T - X_t)}] < \infty,\end{aligned}$$

where we have used Remark 1 in the last step.

To prove that $\gamma \in L^1(\mathbb{R})$, observe that

$$\begin{aligned}\|\gamma\|_{L^1} = \int_{-\infty}^{\infty} |\gamma(k)| \, \mathrm{d}k &= \int_{-\infty}^{\infty} \mathrm{e}^{-\alpha k} \mathbb{E}[(\mathrm{e}^{(X_T - X_t)} - \mathrm{e}^{-k})^+] \mathrm{d}k \\ &= \int_{-\infty}^{\infty} \mathrm{e}^{-\alpha k} \int_{-\infty}^{\infty} (\mathrm{e}^x - \mathrm{e}^{-k})^+ \; \mathbb{Q}(X_T - X_t \in \mathrm{d}x) \mathrm{d}k \\ &= \int_{-\infty}^{\infty} \int_{-x}^{\infty} (\mathrm{e}^{-\alpha k} \mathrm{e}^x - \mathrm{e}^{-(1+\alpha)k}) \mathrm{d}k \; \mathbb{Q}(X_T - X_t \in \mathrm{d}x) \\ &= \frac{1}{\alpha(1+\alpha)} \int_{-\infty}^{\infty} \mathrm{e}^{(1+\alpha)x} \mathbb{Q}(X_T - X_t \in \mathrm{d}x) \\ &= \frac{1}{\alpha(1+\alpha)} \mathbb{E}[\mathrm{e}^{(1+\alpha)(X_T - X_t)}] < \infty.\end{aligned}$$

Note that interchanging the integrals is justified by Tonelli's Theorem, since all involved functions are non-negative. □

The static hedging approach is essentially based on Fourier transform techniques. The Fourier transform of a function f and its inverse transform are given by

$$\mathfrak{F}(f)(\xi) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\xi x} f(x) \; \mathrm{d}x, \qquad \mathfrak{F}^{-1}(f)(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}x\xi} f(\xi) \; \mathrm{d}\xi. \tag{5}$$

We write $C_c(\mathbb{R})$ for the space of continuous functions with compact support, and we introduce the space of functions having Fourier transforms in $C_c(\mathbb{R})$, that is, bandlimited functions with continuous Fourier transform.

Definition 1. Define $\mathfrak{B} \subset L^2(\mathbb{R})$ to be the set of functions with continuous Fourier transforms having compact support, i.e. $\mathfrak{B} = \{f \in L^2 : \mathfrak{F}(f) \in C_c(\mathbb{R})\}$.

Proposition 2. *$\mathfrak{B}$ is dense in $C_c(\mathbb{R})$ with respect to the supremum norm.*

Proof. Apply the Stone-Weierstrass theorem for locally compact spaces, see e.g. Corollary V.8.3 of Conway (1990). □

Proposition 3. *$\mathfrak{F}(\gamma)$ is well defined, bounded, continuous and $|\mathfrak{F}(\gamma)(\xi)| > 0$ holds for any $\xi \in \mathbb{R}$.*

Proof. Since $\gamma \in L^1(\mathbb{R})$ by Proposition 1, we can use Fubini's Theorem to get

$$\begin{aligned}\mathfrak{F}(\gamma)(\xi) &= \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\xi k} \gamma(k) \mathrm{d}k \\ &= \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\xi k} \mathrm{e}^{-\alpha k} \int_{-\infty}^{\infty} (\mathrm{e}^x - \mathrm{e}^{-k})^+ \; \mathbb{Q}(X_T - X_t \in \mathrm{d}x) \; \mathrm{d}k \\ &= \int_{-\infty}^{\infty} \int_{-x}^{\infty} (\mathrm{e}^{(\mathrm{i}\xi - \alpha)k} \mathrm{e}^x - \mathrm{e}^{(\mathrm{i}\xi - 1 - \alpha)k}) \; \mathrm{d}k \; \mathbb{Q}(X_T - X_t \in \mathrm{d}x) \\ &= \frac{1}{(\alpha - \mathrm{i}\xi)(1 + \alpha - \mathrm{i}\xi)} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}(\xi + (1+\alpha)\mathrm{i})x} \mathbb{Q}(X_T - X_t \in \mathrm{d}x) \\ &= \frac{1}{(\alpha - \mathrm{i}\xi)(1 + \alpha - \mathrm{i}\xi)} \varphi_{X_T - X_t}(-\xi - (1+\alpha)\mathrm{i}),\end{aligned} \tag{6}$$

where $\varphi_{X_T-X_t}$ denotes the characteristic function of the random variable $X_T - X_t$. Note that $\varphi_{X_T-X_t}(-\xi-(1+\alpha)\mathrm{i})$ is well defined (using Assumption 1 and e.g. Theorem 4.1 of Lee (2004)) and, since X_t is an additive process, we have

$$\varphi_{X_T-X_t}(z) = \exp\Big\{\int_t^T a(s)\mathrm{d}s\ \Big(\mu\mathrm{i}z - \sigma^2 z^2/2 + \int_{-\infty}^{\infty}(\mathrm{e}^{\mathrm{i}zx} - 1 - \mathrm{i}zx\mathbf{1}_{\{|x|\le 1\}})\nu(\mathrm{d}x)\Big)\Big\}. \tag{7}$$

Finally, as a direct consequence of Assumption 1, the exponent of the right hand side is finite for any $z \in \mathbb{C}$ with $z + (1+\alpha)\mathrm{i} \in \mathbb{R}$, and hence $\mathfrak{F}(\gamma)(\xi) \neq 0$ for all $\xi \in \mathbb{R}$. □

Remark 2. For any $0 < \beta \le \alpha$, the function

$$\gamma(k) = \mathrm{e}^{-\beta k}\mathbb{E}\left[(\mathrm{e}^{X_T-X_t} - \mathrm{e}^{-k})^+\right] = \mathrm{e}^{-\beta k}\,\mathrm{C}(X_t, \mathrm{e}^{-k+X_t})\mathrm{e}^{-X_t}$$

also fulfills all statements of the current section.

3 Existence of approximating hedge portfolios

Now consider a general payoff function $p(X_t)$ to be statically hedged with European calls maturing at $T > t$. The first question to consider is the existence of such a strategy. In general, a hedging strategy may fail to exist and there may even be no acceptable approximating hedging strategy.

The main results in this section establish existence of feasible hedging strategies involving European calls converging to the payoff $p(X_t)$. More specifically, for general payoff functions p, Theorem 1 states almost sure convergence of the hedge. If the payoff function p fulfills some integrability assumptions, Theorem 2 asserts convergence in L^1, which is more useful in practice.

We start with some preliminary statements. The proof of the following Lemma illustrates how a hedge portfolio is determined via the Fourier approach, although the conditions on the payoff function are very restrictive excluding many common payoffs found in practice, such as for example call options.

Lemma 1. *Let p be a continuous payoff function with the property that*

$$e^{-(1+\alpha)x}p(x) \in \mathfrak{B}. \tag{8}$$

There exists a sequence of densities $g_n \in L^1(\mathbb{R}^+)$, such that

$$\lim_{n\to\infty}\mathbb{E}\left[\left|p(X_t) - \int_0^\infty \mathrm{C}(X_t, K)g_n(K)\mathrm{d}K\right|\right] = 0, \tag{9}$$

i.e., the value of the hedging portfolio $\int_0^\infty \mathrm{C}(X_t, K)g_n(K)\mathrm{d}K$ implied by the function g_n converges in mean to the payoff $p(X_t)$.

Proof. Let us start with some informal and heuristic derivations. We would like to find a function g such that

$$p(x) = \int_0^\infty \mathrm{C}(x, K)g(K)\mathrm{d}K. \tag{10}$$

By using

$$\mathrm{C}(x, \mathrm{e}^k) = \mathrm{e}^{(1+\alpha)x}\mathrm{e}^{-\alpha k}\gamma(x-k), \tag{11}$$

and rewriting the term on the right hand side we obtain

$$\int_0^\infty \mathrm{C}(x,K)g(K)\;dK = \int_{-\infty}^{\infty} \mathrm{C}(x,\mathrm{e}^k)g(\mathrm{e}^k)\,\mathrm{e}^k\mathrm{d}k$$
$$= \mathrm{e}^{(1+\alpha)x}\int_{-\infty}^{\infty} \gamma(x-k)g(\mathrm{e}^k)\mathrm{e}^{(1-\alpha)k}\mathrm{d}k.$$

Defining $\tilde{g}(k) = g(\mathrm{e}^k)\mathrm{e}^{(1-\alpha)k}$, Equation (10) is equivalent to the convolution equation

$$\mathrm{e}^{-(1+\alpha)x}p(x) = \gamma * \tilde{g}(x). \tag{12}$$

Defining $\tilde{p}(x) = \mathrm{e}^{-(1+\alpha)x}p(x)$, the Fourier transform of $\tilde{p}$ is given by

$$\mathfrak{F}(\tilde{p}) = \mathfrak{F}(\gamma)\mathfrak{F}(\tilde{g}). \tag{13}$$

Fourier inversion yields

$$\tilde{g}(k) = \mathfrak{F}^{-1}\left(\mathfrak{F}(\tilde{p})/\mathfrak{F}(\gamma)\right)(k)$$

and subsequently we obtain

$$g(K) = K^{\alpha-1}\mathfrak{F}^{-1}\left(\mathfrak{F}(\tilde{p})/\mathfrak{F}(\gamma)\right)(\ln(K)).$$

As stated before this derivation is heuristic and it is a-priori not clear that all operations are well defined. Thus we proceed backwards step by step, and show that all steps are justified.

First note that due to Proposition 3 the Fourier transform $\mathfrak{F}(\gamma)$ is a bounded continuous function with $|\mathfrak{F}(\gamma)(\xi)| > 0$. Also, by Condition (8) we have $\mathfrak{F}(\tilde{p}) \in C_c(\mathbb{R})$. Hence $\mathfrak{F}(\tilde{p})/\mathfrak{F}(\gamma) \in C_c(\mathbb{R})$ and its Fourier inversion $\tilde{g} = \mathfrak{F}^{-1}\left(\mathfrak{F}(\tilde{p})/\mathfrak{F}(\gamma)\right)$ is well defined, continuous and in $L^2(\mathbb{R})$.

With this definition of $\tilde{g}$ we have for all $x \in \mathbb{R}$

$$\mathrm{e}^{-(1+\alpha)x}p(x) = \int_{-\infty}^{\infty} \gamma(x-k)\tilde{g}(k)\mathrm{d}k$$

and subsequently

$$p(x) = \int_{-\infty}^{\infty} \mathrm{C}(x,\mathrm{e}^k)\tilde{g}(k)\mathrm{e}^{\alpha k}\mathrm{d}k = \int_0^\infty \mathrm{C}(x,K)g(K)\mathrm{d}K,$$

where $g(K) = \tilde{g}(\ln(K))K^{\alpha-1}$. Thus the derivation can be made rigourous and we found a well-defined function g satisfying (10).

However, the function g will in general not be in $L^1(\mathbb{R}^+)$; in fact, we only know that $\tilde{g}$ is a continuous $L^2(\mathbb{R})$ function. But for

$$g_n(K) = g(K)\mathbf{1}_{\{K \le \mathrm{e}^n\}}$$

we obtain, using the Hölder inequality,

$$\|g_n(K)\|_{L^1(\mathbb{R}^+)} = \int_{-\infty}^{\infty} |\tilde{g}(k)|\mathrm{e}^{\alpha k}\mathbf{1}_{\{k\le n\}}\;dk \le \|\tilde{g}\|_{L^2(\mathbb{R})}\cdot \mathrm{e}^{2\alpha n}/(2\alpha) < \infty.$$

Finally, define

$$p_n(X_t) = \int_0^\infty C(X_t,K)g_n(K)\,\mathrm{d}K.$$

As a consequence of Proposition 1 we have $\gamma \in L^2(\mathbb{R})$ and we conclude

$$\begin{aligned}\mathbb{E}\left[|p(X_t) - p_n(X_t)|\right] &= \mathbb{E}\left[\left|p(X_t) - \int_0^\infty C(X_t, K) g_n(K)\,\mathrm{d}K\right|\right] \\ &= \mathbb{E}\left[\left|p(X_t) - \int_{\mathbb{R}} \tilde{g}(k)\mathbf{1}_{\{k\leq n\}}\gamma(X_t - k)\,\mathrm{e}^{(1+\alpha)X_t}\,\mathrm{d}k\right|\right] \\ &= \mathbb{E}\left[\mathrm{e}^{(1+\alpha)X_t}\left|\tilde{p}(X_t) - \int \tilde{g}(k)\mathbf{1}_{\{k\leq n\}}\gamma(X_t - k)\,\mathrm{d}k\right|\right] \\ &= \mathbb{E}\left[\mathrm{e}^{(1+\alpha)X_t}\left|\int \tilde{g}(k)\mathbf{1}_{\{k> n\}}\gamma(X_t - k)\,\mathrm{d}k\right|\right] \\ &\leq \mathbb{E}\left[\mathrm{e}^{(1+\alpha)X_t}\|\gamma\|_{L^2(\mathbb{R})}\,\|\tilde{g}\mathbf{1}_{\{\cdot>n\}}\|_{L^2(\mathbb{R})}\right] \\ &= \|\gamma\|_{L^2(\mathbb{R})}\,\|\tilde{g}\mathbf{1}_{\{\cdot>n\}}\|_{L^2(\mathbb{R})}\,\mathbb{E}\left[\mathrm{e}^{(1+\alpha)X_t}\right] \to 0 \qquad \text{as } n\to\infty,\end{aligned}$$

where the last statement follows from $\|\tilde{g}\mathbf{1}_{\{\cdot>n\}}\|_{L^2(\mathbb{R})} \to 0$ as $n \to \infty$, since $\tilde{g} \in L^2(\mathbb{R})$. □

As an immediate consequence of Lemma 1 we get the following Corollary.

Corollary 1. *Assume the payoff function p satisfies condition (8). Then for all $\delta > 0$*

$$\lim_{n\to\infty} \mathbb{P}\left[\left|p(X_t) - \int_0^\infty \mathrm{C}(X_t, K) g_n(K)\ \mathrm{d}K\right| > \delta\right] = 0, \tag{14}$$

i.e., the value of the hedging portfolio $\int_0^\infty \mathrm{C}(X_t, K)g_n(K)\mathrm{d}K$ implied by the $L^1(\mathbb{R}^+)$ functions g_n converges in probability to the payoff $p(X_t)$.

Departing from Assumption (8), we now state hedging results for general payoff functions p.

Theorem 1. *For any measurable payoff function p there exists a sequence of $L^1(\mathbb{R}^+)$ functions g_n, $n \geq 1$, such that*

$$\lim_{n\to\infty} \int_0^\infty \mathrm{C}(X_t, K) g_n(K)\mathrm{d}K = p(X_t) \qquad \mathbb{Q} - a.s. \tag{15}$$

Proof. For any $\epsilon > 0$, there exists a constant $\bar{x}$, such that $\mathbb{P}[|X_t| > \bar{x}] < \epsilon/3$. Now let $\delta > 0$ and write $\tilde{\delta} = \delta/\mathrm{e}^{\bar{x}(1+\alpha)}$ and $\tilde{p}(x) = \mathrm{e}^{-(1+\alpha)x}p(x)$. By Lusin's Theorem (see e.g. Theorem 2.24 of Rudin (1987)) and using the fact that $\mathfrak{B}$ is dense in $C_c(\mathbb{R})$, there exists some $\tilde{h} \in \mathfrak{B}$ such that

$$\mathbb{Q}\left[|\tilde{h}(X_t) - \tilde{p}(X_t)|\mathbf{1}_{\{|X_t|<\bar{x}\}} > \tilde{\delta}/2\right] < \epsilon/3.$$

Note that $h(x) = \mathrm{e}^{(1+\alpha)x}\tilde{h}(x)$ satisfies Condition (8) and hence by Corollary 1 there is some function $g \in L^1(\mathbb{R}^+)$ such that

$$\mathbb{Q}\left[\left|h(X_t) - \int_0^\infty \mathrm{C}(X_t, K)g(K)\mathrm{d}K\right| > \delta/2\right] < \epsilon/3.$$

Then

$$\begin{aligned}\mathbb{Q}\left[\left|p(X_t)-\int_0^\infty \mathrm{C}(X_t,K)g(K)\mathrm{d}K\right|>\delta\right]\\ &< \epsilon/3+\mathbb{Q}\left[\left|p(X_t)-\int_0^\infty \mathrm{C}(X_t,K)g(K)\mathrm{d}K\right|\mathbf{1}_{\{|X_t|<\bar{x}\}}>\delta\right]\\ &< \epsilon/3+\mathbb{Q}\left[\mathrm{e}^{(1+\alpha)X_t}\left|\tilde{p}(X_t)-\tilde{h}(X_t)\right|\mathbf{1}_{\{|X_t|<\bar{x}\}}>\delta/2\right]\\ &\quad+\mathbb{Q}\left[\left|h(X_t)-\int_0^\infty \mathrm{C}(X_t,K)g(K)\mathrm{d}K\right|>\delta/2\right]\\ &< \epsilon.\end{aligned}$$

Thus there is a sequence of $g_n \in L^1(\mathbb{R})$ such that the value of the hedge $\int_0^\infty \mathrm{C}(X_t,K)g_n(K)\mathrm{d}K$ converges in probability to the payoff $p(X_t)$. Finally there is a subsequence for which the convergence is almost sure. □

Theorem 2. *Suppose the measurable payoff function p satisfies the condition*

$$\mathbb{E}[|p(X_t)|]<\infty.$$

Then there exists a sequence of $L^1(\mathbb{R}^+)$ functions g_n, $n\geq 1$, such that

$$\lim_{n\to\infty}\mathbb{E}\left[\left|p(X_t)-\int_0^\infty \mathrm{C}(X_t,K)g_n(K)\mathrm{d}K\right|\right]=0. \tag{16}$$

Proof. First note that μ, defined by

$$\mu(A)=\int_A \mathrm{e}^{(1+\alpha)x}\mathbb{Q}(X_t\in \mathrm{d}x),\quad A\in\mathcal{B}(\mathbb{R}),$$

is a finite measure on $\mathbb{R}$ by Assumption 1. Let $\epsilon>0$. Since $\mathrm{e}^{-(1+\alpha)x}p(x)$ is in $L^1_\mu(\mathbb{R})$, there exists some $\tilde{h}\in\mathfrak{B}$ such that

$$\mathbb{E}\left[\mathrm{e}^{(1+\alpha)X_t}\left|\mathrm{e}^{-(1+\alpha)X_t}p(X_t)-\tilde{h}(X_t)\right|\right]=\int\left|\mathrm{e}^{-(1+\alpha)x}p(x)-\tilde{h}(x)\right|\mu(\mathrm{d}x)<\epsilon/2,$$

since $\mathfrak{B}$ is dense in $C_c(\mathbb{R})$ and $C_c(\mathbb{R})$ is dense in $L^1_\mu(\mathbb{R})$. By Lemma 1 there exists some $g_\epsilon\in L^1(\mathbb{R}^+)$, for which

$$\mathbb{E}\left[\left|\mathrm{e}^{(1+\alpha)X_t}\tilde{h}(X_t)-\int_0^\infty \mathrm{C}(X_t,K)g_\epsilon(K)\mathrm{d}K\right|\right]<\epsilon/2.$$

Summarizing we find

$$\begin{aligned}\mathbb{E}\left[\left|p(X_t)-\int_0^\infty \mathrm{C}(X_t,K)g_\epsilon(K)\mathrm{d}K\right|\right]&\leq \mathbb{E}\left[\mathrm{e}^{(1+\alpha)X_t}\left|\mathrm{e}^{-(1+\alpha)X_t}p(X_t)-\tilde{h}(X_t)\right|\right]\\ &\quad+\mathbb{E}\left[\left|\mathrm{e}^{(1+\alpha)X_t}\tilde{h}(X_t)-\int_0^\infty \mathrm{C}(X_t,K)g_\epsilon(K)\mathrm{d}K\right|\right]\\ &<\epsilon.\end{aligned}$$

Since ϵ was arbitrary this completes the proof. □

Remark 3. The assumption $\mathbb{E}[|p(X_t)|]<\infty$ is only slightly stronger than assuming the contract can be given a finite price. So it is not at all stringent.

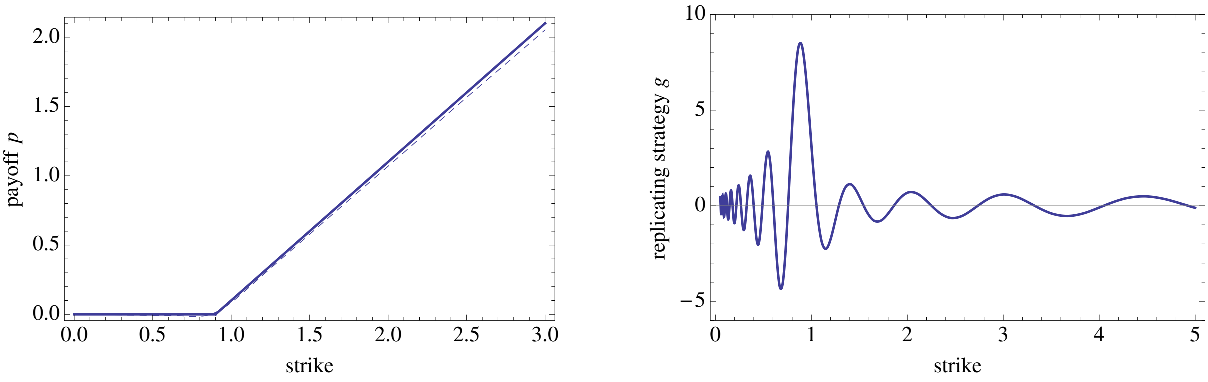

Figure 1: Static replication of a call option maturing in 0.5 years with call options maturing in one year. The replicating strategy is determined via the Nyquist-Shannon Theorem. Left: payoff p (dashed line shows true payoff), right: replicating strategy g.

4 Constructing approximating hedge portfolios

Except for the case of payoffs satisfying Condition (8), the results of the previous section merely state the existence of an approximating hedge portfolio. In this section we want to get these results to actually work. Directly implementing the Fourier inversion approach used in Lemma 1 for a payoff that does not satisfy the conditions and yields highly unstable and oscillatory hedging strategies.[1]

A natural approach to deriving concrete approximating hedging strategies in the sense of Theorem 2 is to approximate a general payoff by a payoff satisfying the conditions of Lemma 1. This can be achieved for example by applying the *Nyquist-Shannon Theorem*, which amounts to decomposing a bandlimited function in terms of sinc functions, which can be hedged perfectly using the setup of Lemma 1. Since payoffs encountered in finance are not in general bandlimited, the approximation essentially consists of artificially bandlimiting the payoff. However, depending on the degree of precision desired, this approach may still lead to oscillatory hedging strategies. In any case, the approach is somewhat *ad hoc* and does not necessarily yield an optimal result for some given stability criterion. An example in Figure 1 demonstrates the issue. Here, assuming a Black-Scholes model with volatility $\sigma = 0.16836$, a call option with strike $K = 0.9$ maturing in 0.5 years is hedged using call options maturing in one year. The replicating strategy, shown on the right, illustrates two rather undesirable traits: first, large numbers of options at individual strikes need to be traded, and second, options at unusually high strikes need to be traded. Details on the calculation of the replicating strategy are given in Appendix A.

To overcome these problems, and to attain optimal solutions in a certain sense, it turns out that regularisation techniques lead to conrete workable realisations of approximating hedging strategies. Regularisation comprises a number of techniques to deal with the instability of solutions to ill-posed inverse problems (in the sense of Hadamard, 1902). In fact, the problem we are facing is an inverse problem, as can be seen as follows. Assume for the moment that the asset price model is a plain Black-Scholes model. Then the hedging portfolio of a payoff can be understood as the solution (at time 0) of the well-known parabolic heat equation. The problem one is facing is that the equation has to be solved "in the wrong direction", i.e., the Kolmogorov forward equation has to be solved with a final condition instead of an initial condition. This, in

[1] Even for a payoff as simple as a call option, we obtain numerical values for g in the order of 10^8, and it is not clear how much of this quantity is due to numerical imprecision.

particular, qualifies the problem as an ill-posed inverse problem, that needs some regularization to be stabilized.

Let us shortly revise the goal: according to Equation (12) for some payoff function the hedging problem amounts to solving for $\tilde{g}$ the equation

$$\tilde{p}(x) = \mathrm{e}^{-(1+\alpha)x} p(x) = \gamma * \tilde{g}(x). \tag{17}$$

On the right hand side there is a convolution operator with the kernel γ. This kind of problem arises in many fields, such as image recognition, and is well studied. Solutions of this type of problem are typically unstable in the sense that a small perturbation of the output $\tilde{p}$ leads to a large change in the solution $\tilde{g}$. Regularisation, such as *Tikhonov regularization*, arises as a standard method to handle such problems, see e.g. Bertero and Boccacci (1998).

Consider therefore the regularized solution for $\tilde{g}$ given by

$$\tilde{g}_\mu(k) = \left(\mathfrak{F}^{-1}\left(\frac{\hat{\gamma}^*}{|\hat{\gamma}|^2 + \mu}\right) * \tilde{p}\right)(k), \tag{18}$$

where $\mu > 0$ is the regularization parameter and $\hat{\gamma} = \mathfrak{F}(\gamma)$ is the Fourier transform of γ.

It can be shown (cf. Section 5.1 of Bertero and Boccacci (1998)) that (18) solves the optimization problem

$$\min_{\tilde{g}} \|\gamma * \tilde{g} - \tilde{p}\|_2^2 + \mu \|\tilde{g}\|_2^2, \tag{19}$$

which trades off the error of the solution against the norm of the solution.

For the sequence of regularization parameters $(1/n)_{n\in\mathbb{N}}$, we prove that the payoff implied by the corresponding hedging portfolio converges to the payoff p, which we wish to hedge. Recall that we always suppose that Assumption 1 is fulfilled and α is the parameter from (3). It should further be noted that the assumptions in Theorem 3 below are fulfilled for most examples relevant in practice and are thus not very restrictive.

Theorem 3. *Let p be some payoff function. Suppose there exists $0 < \beta < \alpha$ such that*

(i) $\tilde{p} \in L^2(\mathbb{R})$, *where* $\tilde{p}(x) = e^{-(1+\beta)x} p(x)$,

(ii) X_t *has a continuous density* q *with respect to the Lebesgue measure,* $\mathbb{Q}(X_t \in \mathrm{d}x) = q(x)\mathrm{d}x$, *and the density satisfies* $e^{(1+\beta)x} q(x) \in L^2(\mathbb{R})$,

(iii) $\mathbb{E}[|p(X_t)|] < \infty$.

Let[2]

$$\gamma(k) = e^{-\beta k}\mathbb{E}\left[(e^{X_T - X_t} - e^{-k})^+\right] = e^{-\beta k}\,\mathrm{C}(X_t, e^{-k+X_t})e^{-X_t}$$

and define

$$g_n(K) = K^{\beta-1}\mathbf{1}_{[0,e^n)}(K) \cdot \tilde{g}_{1/n}(\ln K), \tag{20}$$

where $\tilde{g}_{1/n}$ *is given by Equation* (18)*. Then* (g_n) *is a sequence of* $L^1(\mathbb{R}^+)$ *functions with*

$$\lim_{n\to\infty} \mathbb{E}\left[\left|p(X_t) - \int_0^\infty \mathrm{C}(X_t, K) g_n(K)\mathrm{d}K\right|\right] = 0. \tag{21}$$

[2]Compare Remark 2.

Proof. First we prove that g_n is in fact a sequence of $L^1(\mathbb{R}^+)$ functions. Indeed, by applying Hölder's inequality,

$$\begin{aligned}\int_0^\infty |g_n(K)|\mathrm{d}K &= \int_{-\infty}^\infty \mathrm{e}^{\beta k}\mathbf{1}_{(-\infty,n)}(k)|(\tilde{g}_{1/n}(k)|\mathrm{d}k \\ &\leq \|\mathrm{e}^{\beta k}\mathbf{1}_{(-\infty,n)}(k)\|_{L^1(\mathbb{R})}\|\tilde{g}_{1/n}\|_{L^\infty(\mathbb{R})} < \infty,\end{aligned}$$

where the last line follows from Young's inequality,

$$\begin{aligned}\|\tilde{g}_{1/n}\|_{L^\infty(\mathbb{R})} &= \left\|\mathfrak{F}^{-1}\left(\frac{\hat{\gamma}^*}{|\hat{\gamma}|^2+1/n}\right) * \tilde{p}\right\|_{L^\infty(\mathbb{R})} \leq \frac{1}{2\pi}\left\|\frac{\hat{\gamma}^*}{|\hat{\gamma}|^2+1/n}\right\|_{L^2(\mathbb{R})}\|\tilde{p}\|_{L^2(\mathbb{R})} \\ &\leq \frac{n}{2\pi}\|\hat{\gamma}^*\|_{L^2(\mathbb{R})}\|\tilde{p}\|_{L^2(\mathbb{R})} = n\|\gamma\|_{L^2(\mathbb{R})}\|\tilde{p}\|_{L^2(\mathbb{R})} < \infty \end{aligned} \tag{22}$$

and Proposition 1, Remark 2.
Now observe that

$$\begin{aligned}\mathbb{E}\left[\left|p(X_t) - \int_0^\infty \mathrm{C}(X_t,K)g_n(K)\mathrm{d}K\right|\right] &= \mathbb{E}\left[\mathrm{e}^{(1+\beta)X_t}\left|\tilde{p}(X_t) - \int_{-\infty}^n \gamma(X_t-k)\tilde{g}_{1/n}(k)\mathrm{d}k\right|\right] \\ &\leq \mathbb{E}\left[\mathrm{e}^{(1+\beta)X_t}\left|\int_n^\infty \gamma(X_t-k)\tilde{g}_{1/n}(k)\mathrm{d}k\right|\right] \\ &\quad + \mathbb{E}\left[\mathrm{e}^{(1+\beta)X_t}\left|\tilde{p}(X_t) - \int_{-\infty}^\infty \gamma(X_t-k)\tilde{g}_{1/n}(k)\mathrm{d}k\right|\right].\end{aligned} \tag{23}$$

By Proposition 1 and Remark 2 we have $\|\mathrm{e}^{(\beta-\alpha)k}\gamma(k)\|_{L^\infty(\mathbb{R})} = c < \infty$ and thus applying Fubini's Theorem yields

$$\begin{aligned}&\mathbb{E}\left[\left|\mathrm{e}^{(1+\beta)X_t}\int_n^\infty \gamma(X_t-k)\tilde{g}_{1/n}(k)\mathrm{d}k\right|\right] \\ &\quad\leq \int_n^\infty |\tilde{g}_{1/n}(k)|\mathbb{E}\left[\mathrm{e}^{(1+\beta)X_t}\gamma(X_t-k)\right]\mathrm{d}k \\ &\quad= \int_n^\infty |\tilde{g}_{1/n}(k)|\mathbb{E}\left[\mathrm{e}^{(1+\beta)X_t}\mathrm{e}^{(\alpha-\beta)(X_t-k)}\mathrm{e}^{(\beta-\alpha)(X_t-k)}\gamma(X_t-k)\right]\mathrm{d}k \\ &\quad\leq c\int_n^\infty |\tilde{g}_{1/n}(k)|\mathrm{e}^{(\beta-\alpha)k}\mathbb{E}\left[\mathrm{e}^{(1+\alpha)(X_t)}\right]\mathrm{d}k \\ &\quad\leq c\cdot\mathbb{E}\left[\mathrm{e}^{(1+\alpha)X_t)}\right]\ \|\mathrm{e}^{(\beta-\alpha)\cdot}\mathbf{1}_{[n,\infty)}\|_{L^1(\mathbb{R})}\ \|\tilde{g}_{1/n}\|_{L^\infty(\mathbb{R})} \\ &\quad\leq \frac{c}{\alpha-\beta}\cdot\mathbb{E}\left[\mathrm{e}^{(1+\alpha)X_t)}\right]\ \|\tilde{g}_{1/n}\|_{L^\infty(\mathbb{R})}\ \mathrm{e}^{(\beta-\alpha)n} \\ &\quad\leq \frac{c}{\alpha-\beta}\cdot\mathbb{E}\left[\mathrm{e}^{(1+\alpha)X_t)}\right]\ n\|\gamma\|_{L^2(\mathbb{R})}\|\tilde{p}\|_{L^2(\mathbb{R})}\mathrm{e}^{(\beta-\alpha)n} \ \to 0,\ \text{as } n\to\infty,\end{aligned}$$

where we have used (22) and Assumption 1.

For the second term on the right-hand side of (23) observe that

$$\begin{aligned}&\mathbb{E}\left[\mathrm{e}^{(1+\beta)X_t}\left|\tilde{p}(X_t) - \int_{-\infty}^\infty \gamma(X_t-k)\tilde{g}_{1/n}(k)\mathrm{d}k\right|\right] \\ &\qquad= \mathbb{E}\left[\mathrm{e}^{(1+\beta)X_t}\left|\tilde{p}(X_t) - \left(\mathfrak{F}^{-1}\left(\frac{|\hat{\gamma}|^2}{|\hat{\gamma}|^2+1/n}\right) * \tilde{p}\right)(X_t)\right|\right].\end{aligned}$$

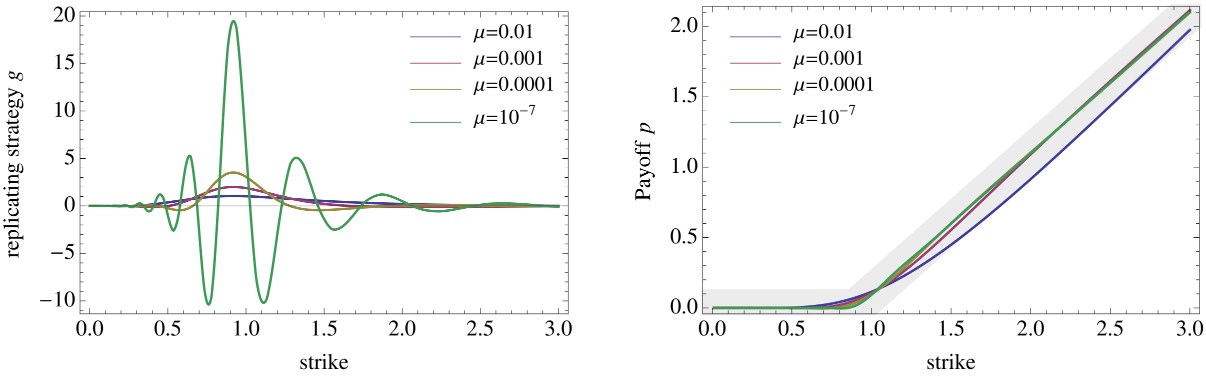

Figure 2: Static replication of a call option maturing in 0.5 years with call options maturing in one year. Replicating strategies are regularised with parameter μ.

By Assumption (ii) the law of X_t under $\mathbb{Q}$ admits a continuous density q. Writing $\hat{\tilde{p}} = \mathfrak{F}(\tilde{p})$ we conclude

$$
\begin{aligned}
&\mathbb{E}\left[\mathrm{e}^{(1+\beta)X_t}\left|\tilde{p}(X_t) - \int_{-\infty}^{\infty}\gamma(X_t-k)\tilde{g}_{1/n}(k)\mathrm{d}k\right|\right] \\
&\qquad = \int_{-\infty}^{\infty}\mathrm{e}^{(1+\beta)x}\left|\tilde{p}(x) - \left(\mathfrak{F}^{-1}\left(\frac{|\hat{\gamma}|^2}{|\hat{\gamma}|^2+1/n}\right)*\tilde{p}\right)(x)\right| q(x)\mathrm{d}x \\
&\qquad = \int_{-\infty}^{\infty}\mathrm{e}^{(1+\beta)x}\left|\mathfrak{F}^{-1}\left(\hat{\tilde{p}}\right)(x) - \mathfrak{F}^{-1}\left(\frac{|\hat{\gamma}|^2\cdot\hat{\tilde{p}}}{|\hat{\gamma}|^2+1/n}\right)(x)\right| q(x)\mathrm{d}x \\
&\qquad = \int_{-\infty}^{\infty}\mathrm{e}^{(1+\beta)x}\left|\mathfrak{F}^{-1}\left(\frac{\hat{\tilde{p}}/n}{|\hat{\gamma}|^2+1/n}\right)(x)\right| q(x)\mathrm{d}x \\
&\qquad \leq \left\|\mathrm{e}^{(1+\beta)\cdot}q\right\|_{L^2(\mathbb{R})}\left\|\mathfrak{F}^{-1}\left(\frac{\hat{\tilde{p}}/n}{|\hat{\gamma}|^2+1/n}\right)\right\|_{L^2(\mathbb{R})} \\
&\qquad = \left\|\mathrm{e}^{(1+\beta)\cdot}q\right\|_{L^2(\mathbb{R})}\left\|\frac{\hat{\tilde{p}}/n}{|\hat{\gamma}|^2+1/n}\right\|_{L^2(\mathbb{R})}.
\end{aligned}
$$

Since $\hat{\tilde{p}}/(n|\hat{\gamma}|^2+1) < \hat{\tilde{p}}$ and $\hat{\tilde{p}} \in L^2(\mathbb{R})$ by Assumption (i), Lebesgue's Dominated Convergence Theorem finally completes the proof. □

As an example, consider again the call option maturing in 0.5 years that is hedged with options maturing in one year. The underlying asset price process follows a Black-Scholes model with volatility $\sigma = 0.16836$. Figure 2 shows the replicating strategies and the resulting payoffs for various regularisation parameters. As the regularisation parameter increases, both the number of trades in the replicating portfolio and the quality of the strategy decrease.

Remark 4. Consider a payoff $p(x)$ that can be represented as linear combination ("mixture") of parameterized payoffs $p_{\tilde{K}}(x)$,

$$p(x) = \int p_{\tilde{K}}(x)\mu(\mathrm{d}\tilde{K}), \tag{24}$$

with weights $\mu(\mathrm{d}\tilde{K})$. We assume that the conditions of Theorem 3 are satisfied for p and the payoffs $p_{\tilde{K}}$. Denote by g_n and $g_{n,\tilde{K}}$ the static hedge functions from Theorem 3 for p and $p_{\tilde{K}}$, respectively. Then, by linearity and under some technical conditions needed to apply Fubinis' Theorem, the static hedge functions are related by

$$g_n(K) = \int g_{n,\tilde{K}}(K)\mu(\mathrm{d}\tilde{K}).$$

A well-known example for a representation of the form (24) is the representation of sufficiently smooth payoffs as superposition of call options, $p_{\tilde{K}}(x) = (\mathrm{e}^x - \tilde{K})^+$, cf. Carr and Madan (1998). Denote $\bar{p}(s) = p(\ln s)$ and assume that this function is well-defined and sufficiently smooth for $s \in [0, \infty)$. Then

$$p(x) = \bar{p}(\mathrm{e}^x) = \bar{p}(0) + \bar{p}'(0)\mathrm{e}^x + \int_0^\infty (\mathrm{e}^x - \tilde{K})^+ \bar{p}''(\tilde{K})\mathrm{d}\tilde{K},$$

and the static hedging strategy involves $\bar{p}(0)$ zero bonds maturing at time t, $\bar{p}'(0)$ shares of the underlying security S_t and static hedge positions,

$$g_n(K) = \int g_{n,\tilde{K}}(K)\bar{p}''(\tilde{K})\mathrm{d}\tilde{K},$$

in call options with strike K maturing at time T.

5 Approximations with a finite number of hedging instruments

In practice, only a finite – possibly small – number of hedging instruments is available. This requires that the approximate replicating portfolio of Theorem 3, given by

$$p(X_t) \approx \int_0^\infty C(X_t, K)\, g_n(K)\, \mathrm{d}K, \tag{25}$$

with g_n given by Equation (20), be suitably approximated (here, we assume that the parameter n, which determines the degree of regularisation, is fixed in advance, but we shall discuss later how this can be determined in practice).

We investigate approximation methods: discretising the right-hand side of (25) via Gauss quadrature, see Section 5.1, and a state-space discretisation, which allows for determining the optimal hedge by solving a linear system, see Section 5.1.

5.1 Discretisation with Gauss quadrature

We approximate the right-hand side of (25) by a weighted sum of m evaluations,

$$\int_0^\infty C(X_t, K)\, g_n(K)\, \mathrm{d}K \approx \sum_{i=1}^m w_i\, C(X_t, K_i)\, g_n(K_i),$$

where the weights $w_1, \ldots, w_m$ and strikes $K_1, \ldots, K_m$ are chosen according to some quadrature rule. In our example we shall use the Gauss-Legendre quadrature method (see (Abramowitz and Stegun, 1972), Section 25.4.29, p. 887). Here, K_i is the i-th zero of the Legendre polynomial P_m of m-th degree and the related weights are $w_i = 2/\left[(1 - K_i^2)P_m'(K_i)^2\right]$, $i = 1, \ldots, m$.

Of course, choosing the strikes within the method is feasible only if there is a sufficiently large universe of hedging instruments available. If the number of hedging instruments is small, then other discretisation methods may be used or the state-space discretisation below may be considered.

5.2 Linear system via state-space discretisation

By assuming a finite number of hedging instruments and by further discretising the state space, the static hedging problem reduces to solving a linear system. Solving discrete inverse problems is a well-studied problem, in particular with respect to the number of regularization techniques available, see e.g. Björck (1996), Hansen (1998), Hansen (2010) or (Press et al., 1992, Chapter 18). Thus, instead of starting with the approximate hedge (25), one may directly solve for g the following linear system

$$\begin{pmatrix} C(x_1; K_1) & \cdots & C(x_1; K_m) \\ \vdots & \ddots & \vdots \\ C(x_q; K_1) & \cdots & C(x_q; K_m) \end{pmatrix} \begin{pmatrix} g(K_1) \\ \vdots \\ g(K_m) \end{pmatrix} = \begin{pmatrix} p(x_1) \\ \vdots \\ p(x_q) \end{pmatrix}$$

with m hedging instruments and q states $x_1, \ldots, x_q$. We shall often use the short-hand notation

$$Cg = p,$$

where $C \in \mathbb{R}^{q \times m}$, $g \in \mathbb{R}^m$, $p \in \mathbb{R}^q$.

When $m \ll q$, that is, the number of hedging instruments is much smaller than the number of targeted payoff states, the system is overdetermined and in general no solution exists. The least-squares solution of the linear system is the solution to the following minimization problem:

$$\min_g \|Cg - p\|_2.$$

Since the solution to the above problem might not be unique, one defines to the so-called *general linear least squares problem*

$$\min_{g \in \mathcal{S}} \|g\|_2, \text{ where } \mathcal{S} = \{g \in \mathbb{R}^n \| \|Cg - p\|_2 = \min\}. \tag{26}$$

This problem yields a unique solution and can be efficiently solved using standard methods such as singular value decomposition (SVD) of C to determine the so-called *pseudoinverse (Moore-Penrose inverse)* of C, see e.g. Section 1.2 of Björck (1996).

As in the continuous case, the hedge g is the solution to an ill-posed inverse problem and as such will produce high numbers of traded options to hedge the payoff. This leads again to regularization methods, such as Tikhonov regularisation

$$\min_g \left(\|Cg - p\|_2^2 + \mu \|g\|_2^2 \right), \tag{27}$$

which – depending on the magnitude of the parameter $\mu \in \mathbb{R}$ – gives preference to a solution with a smaller norm. Other regularization methods give preference to "smooth" solutions or constrain the total number of options traded.

6 Examples and empirical results

We shall now put theory to practice and analyze how the results developed perform for different payoffs and in different models. First, we discuss the model setup and data that the examples are based on. Second, we discuss a method of determining the regularisation parameter μ. Next, we compare hedging strategies in different models for a number of payoffs. As model candidates we consider a standard Black-Scholes model and a CGMY model. The CGMY model is more appropriate for capturing the volatility surface present in options markets. Finally, similar to (Carr and Wu, 2013), we compare the hedging performance of a static hedge, when only a finite number of hedging instruments is available, with the performance of a dynamic hedge when the hedging portfolio is rebalanced only a discrete points in time, as would be the case in practice. In addition to the Black-Scholes and CGMY models, we consider a Heston model as a data-generating process. This captures the stochastic volatility that is generally present in markets. Aside from the hedging error due to discretisation, this will generate additional P&L, as the stock price process under the Heston model does not fulfill the additivity assumption and hence does not comply with the framework here.

6.1 Models and data

In our analysis we consider three models: the Black-Scholes model, the Heston model, (Heston, 1993), and the CGMY model, (Carr et al., 2002). The Black-Scholes model and the CGMY model are both exponential processes as specified in (1) and fulfil the assumptions made in the paper. On the other hand, under the Heston model, which is a stochastic volatility model, the asset price process fails to be an additive process. As such, our static hedging approach will not yield perfect hedges, and the purpose of analysis in this case is to study the hedging performance under the realistic assumption of stochastic volatility (see also (Carr and Wu, 2013) who perform a similar analysis). Under the risk-neutral measure $\mathbb{Q}$, the dynamics of the asset price processes are given by

$$\begin{aligned} &\text{BS:} && \mathrm{d}S_t = (r-q)\,S_t\,\mathrm{d}t + \sigma\,S_t\,\mathrm{d}W_t \\ &\text{CGMY:} && \mathrm{d}S_t = (r-q)\,S_{t-}\,\mathrm{d}t + S_{t-}\,\mathrm{d}X_t \\ &\text{Heston:} && \mathrm{d}S_t = (r-q)\,S_t\,\mathrm{d}t + \sqrt{V_t}S_t\,\mathrm{d}W_{1,t}, \quad \text{where} \\ & && \mathrm{d}V_t = \kappa(\theta - V_t)\,\mathrm{d}t + \nu\,\sqrt{V_t}\,\mathrm{d}W_{2,t}, \end{aligned}$$

where W W_1 and W_t are Brownian motions, W_1 and W_2 are correlated with instantaneous correlation ρ, and X is a CGMY process, that is, a Lévy process with characteristic exponent

$$\begin{aligned} \psi_{CGMY}(u;C,G,M,Y) = iu(&-C\Gamma(-Y)((M-1)^Y - M^Y + (G+1)^Y - G^Y)) \\ &+ C\,\Gamma(-Y)((M-iu)^Y - M^Y + (G+iu)^Y - G^Y), \quad u \in \mathbb{R}. \end{aligned}$$

This implies that the characteristic function of $X_T - X_t$ is given by

$$\phi_{X_T - X_t}(u) = \exp(\psi_{CGMY}(u;C,G,M,Y)\,(T-t)).$$

The first term of ψ_{CGMY} implies that S is a martingale under $\mathbb{Q}$, cf. Section 6.2.2 of (Schoutens, 2003).

All models are calibrated to S&P option prices from 15 May 2013. The data involve 77 option prices of various strikes and maturities and calibration is achieved by minimizing the root mean square error of model prices against market prices. The data are given in Appendix B. For the CGMY and Heston models, option prices are calculated using the Fast Fourier transform method from Carr and Madan (1999). The calibrated model parameters are given in Table 1 and the fitted call prices from the models are shown in Figure 7 in Appendix B.

6.2 Choice of regularisation parameter

A concrete realisation of the results developed requires choosing the parameter μ in Equation (18), resp. $1/n$ in Theorem 3. One simple criterion is the so-called *L-curve*, which visualises the trade-off of the quadratic error of the regularized solution $\|\gamma * \tilde{g} - \tilde{p}\|_2^2$ against the regularization constraint $\|\tilde{g}\|_2^2$ in the solution of Equation (19) for various regularization parameters μ. The

Model	Parameters				
Black-Scholes	σ				
	0.16836				
CGMY	C	G	M	Y	
	0.0175119	0.0314313	6.8318104	1.3283701	
Heston	V_0	θ	κ	ν	ρ
	0.013681	0.053318	1.605179	0.590506	-0.620100

Table 1: Model parameters from calibration.

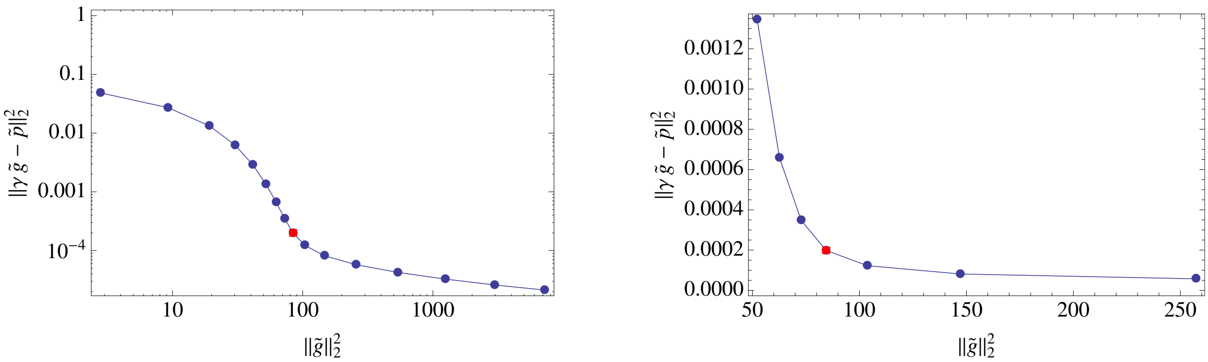

Figure 3: Determination of the regularization parameters using the L-curve. The point marked in red, which corresponds to $\mu = 10^{-9/2}$, is chosen as relaxing either one of the criteria further does not provide a significant improvement in the criterion. Left: log-log scale, right: normal scale.

L-curve provides a convenient method to display the trade-off between the error introduced by regularization, that is, the hedge quality, and the benefit of regularization, which results in a smaller number of options traded for hedging. For details on the L-curve we refer to e.g. Chapter 5 of (Bertero and Boccacci, 1998), Chapter 4 of Hansen (1998) and Hansen (2010).

For the example from Section 4, cf. also Figure 2, the L-curve is shown in Figure 3. A reasonable choice of the regularisation parameter is the point marked in red, as a further improvement in the hedge error requires significantly more traded options, and a further improvement in the number of traded options increases the hedging error significantly.

In practice the number of traded options often matters, which is not appropriately captured by the L-curve criterion, so that in the examples we sometimes increase the regularization parameter (and indicate this) in order to reduce the number of trades.

In order to make the hedge strategies comparable between the Black-Scholes and the CGMY models, we use the same regularization parameters for the CGMY models as determined for the Black-Scholes models.

6.3 Hedge quality for different models

We compare the hedging strategies resulting from Formula (18) for different payoffs in the Black-Scholes model and in the CGMY-model. Static replicating strategies are calculated in three setups: assuming a continuum of hedging instruments, and two strategies based on a finite number of hedging instruments, namely the Gaussian quadrature method from Section 5.1 and the least squares optimisation from Section 5.2. In the discrete cases we assume that 11 hedging instruments with strikes ranging from 0.3 to 2.5 are available. In the Gaussian

Method	Continuous	Quadrature	Least Squares
		Black-Scholes	
No. options	1.70	1.77	1.22
RMSE	0.005973	0.01425	0.01648
Cost of hedge	0.1114	0.1216	0.1441
		CGMY	
No. options	1.46	1.40	0.56
RMSE	0.004740	0.03549	0.01598
Cost of hedge	0.1135	0.1294	0.1444
RMSE (BS)	0.007292	0.0372	0.01707
Cost of hedge (BS)	0.1100	0.1216	0.1413

Table 2: Hedging a call option. The Black-Scholes price of the payoff is 0.1116.

quadrature method the exact strikes are determined by the method, whereas for the least squares optimisation we assume evenly spaced strikes.

6.3.1 Call option

In the first example we replicate a standard call option with strike 0.9 and maturity $t = 0.5$ years. For the continuum of strikes, the regularization parameter is set to $\mu = 10^{-9/2}$, chosen according to the L-curve criterion in the previous section. For the least squares optimisation the regularisation parameter was chosen to be $\mu = 1/2$, again according to the L-curve criterion.[3] Each row in Figure 4 corresponds to one of the three setups discussed above. The individual figures in each row show the replicating portfolios g for both the Black-Scholes and the CGMY models, their respective payoffs and the error in the payoff generated by the replicating portfolio. For the CGMY hedge portfolio we show in addition the hedge error when the underlying stock price follows in fact a Black-Scholes model. This allows to get an idea of the model risk involved. Further information on the static replication strategies, such as the number of traded options to set up the hedge, the root mean square error and the cost of the hedge are given in Table 2. The last two lines again refer to the setup where the stock price process follows a Black-Scholes model, but hedging takes place in a CGMY model. In all cases, the greatest hedge error occurs when the call option expires at-the-money, which has to do with the singularity at this point. The hedging error at this point is greatest for the discrete least squares optimisation; one can see from the payoff that the hedging strategy superhedges the claim when it expires at-the-money. This may also help explain why this strategy is the most expensive strategy (cf. Table 2). The Gaussian quadrature method fails when the option expires strongly in the money, with the loss being greatest for the CGMY method. It is not surprising that the least squares method clearly performs better than the Gaussian quadrature, as the replicating strategy determined via least squares corresponds to a (constrained) optimum. The Gaussian quadrature method provides parameters for numerical integration, but is independent of the actual function to be integrated. In terms of the number of traded options, the least squares method is favourable, although it is by far the most costly method. Finally, it is evident that all hedging strategies are somewhat similar, which in particular demonstrates the relationship between the regularised method with a continuum of strikes, which is a least squares optimum, cf. Equation (19), and the discretised least squares solution.

Note that for the BS-model and the CGMY-model the difference in the hedging portfolios is

[3] Note that the respective regularisation parameters are not directly comparable, which is also the reason for determining each parameter separately.

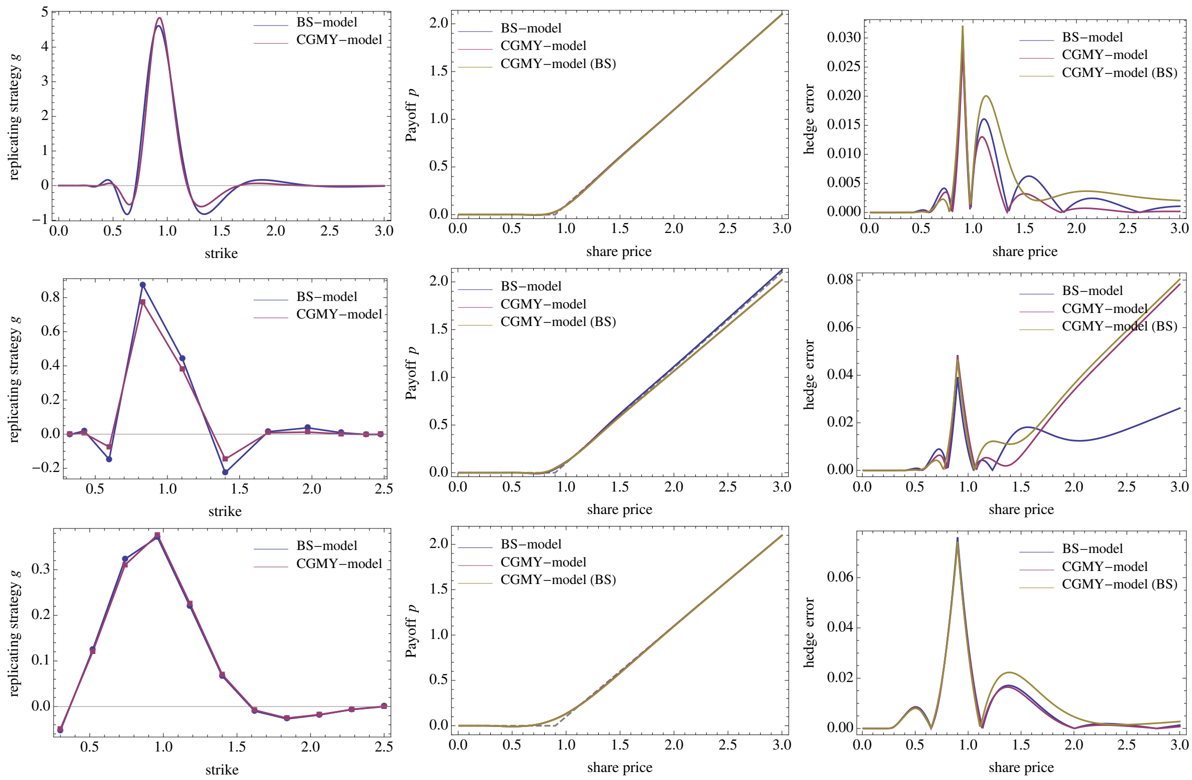

Figure 4: Hedging a call option. Top: continuum of strikes; middle: Gaussian quadrature; bottom: Least squares optimal strategy. Left: Replicating strategies in Black-Scholes and CGMY models. Middle: Payoff from replicating strategies (dashed line shows option payoff). Right: Hedge error for Black-Scholes model, CGMY model and when hedging according to a CGMY model but underlying process follows a Black-Scholes model.

relatively small. This is both surprising as the two models are very different, and, on the other hand very encouraging, since the procedure seems to be fairly robust with respect to model risk.

Finally, it should be noted that the Gaussian quadrature method performs well only if the regularisation parameter is chosen large enough. This has inherently to do with the osciallatory nature of the hedging strategy, which is dampened by regularisation. If this is not sufficiently dampened, then a discretisation at a relatively small number of points produces unreliable results.

6.3.2 Binary option

In the next example we study the replication of a binary option with strike 0.9 and maturity $t = 0.5$. This is a more challenging example, due to the jump at the strike of the option. Figure 5 shows the hedging portfolios g and the respective payoffs at maturity for the Black-Scholes and the CGMY-models. The regularization parameter in the continuous case was chosen to be $\mu = 2 \cdot 10^{-4}$.[4] For the discretised least squares method, the regularisation parameter is chosen to be $\mu = 2^{-3}$.[5] As expected, the performance of the replicating strategies is worse than compared to the call payoff; this is somewhat intrinsic as the Fourier inversion approach performs best for

[4] According to the L-curve criterion one would rather choose $\mu = 10^{-5}$, but the resulting replicating strategy entails trading approximately 50 ATM options and 50 options with a strike of around 1, which is not reasonable in practice.

[5] Again by the L-curve criterion, one would choose $\mu = 2^{-7}$ resulting in a large number of traded options.

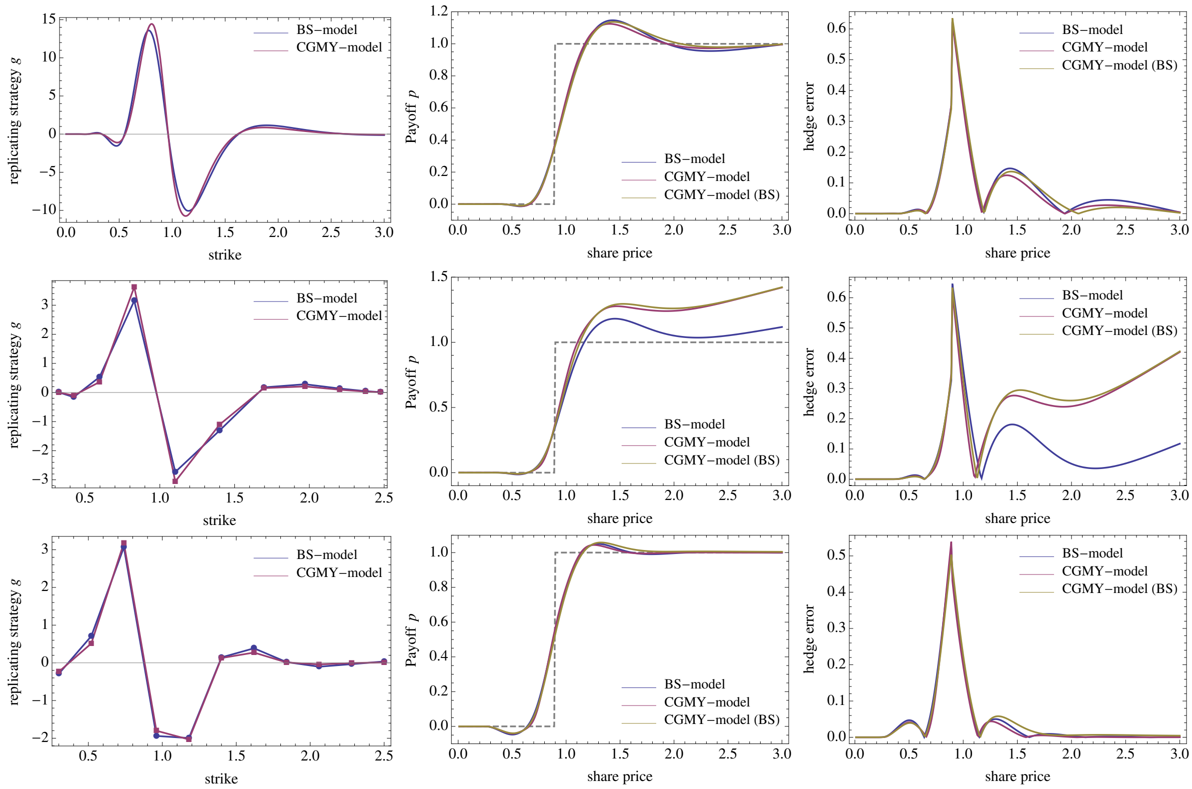

Figure 5: Hedging a binary option. Top: continuum of strikes; middle: Gaussian quadrature; bottom: Least squares optimal strategy. Left: Replicating strategies in Black-Scholes and CGMY models. Middle: Payoff from replicating strategies (dashed line shows option payoff). Right: Hedge error for Black-Scholes model, CGMY model and when hedging according to a CGMY model but underlying process follows a Black-Scholes model.

smooth payoff functions. Here, at-the-money, the error amounts to approximately 60% of the underlying stock price, which corresponds to 75% of the option premium (the option premium is 0.7970 in the Black-Scholes model). Further information on the performance of the hedging strategies is given in Table 3. Again, the least squares optimal solution is favourable over the quadrature method in terms of mean square error, but comes at a higher cost as well.

6.4 Static hedging versus dynamic hedging

We now compare the performance of static hedging and dynamic hedging. This example is similar to the simulation study in (Carr and Wu, 2013). Despite yielding perfect hedges in theory for models such as the Black-Scholes model, both static hedging and dynamic hedging produce only imperfect hedges in practice. For static hedging, the hedge error is related to choosing a regularisation parameter and to the limited number of hedging instruments, whereas for dynamic hedging, the error from hedging is due to discrete rebalancing. Furthermore, for incomplete market models, additional P&L is introduced if a claim cannot be replicated even in theory.

In the following examples we compare the hedging errors from static and dynamic hedging. The models we consider are the Black-Scholes model, the CGMY model and the Heston model. In the Black-Scholes model, the hedging error is caused by the imperfections outlined above. In the CGMY and the Heston models, a further error is introduced when delta-hedging due to

Parameter	Continuous	Quadrature	Least Squares
		Black-Scholes	
No. options	7.86	8.54	8.72
RMSE	0.1300	0.1407	0.1061
Cost of hedge	0.6099	0.6114	0.7395
		CGMY	
No. options	7.66	8.74	4.23
RMSE	0.1219	0.2493	0.1035
Cost of hedge	0.6504	0.7166	0.7742
RMSE (BS)	0.1296	0.2596	0.1061
Cost of hedge (BS)	0.6001	0.6114	0.7251

Table 3: Hedging a binary option. The Black-Scholes price of the payoff is 0.7970.

model incompleteness. Although dynamic delta-hedging may not be the optimal hedge strategy, this is still prevalent in practice, so that a comparison of dynamic and static hedging strategies is of interest from a practitioner's point of view. Static hedges in the Heston model are in addition imperfect as asset returns are not independent as required in our setup. In fact, we use $\mathfrak{F}(\gamma)$ as in Equation (6) to determine the hedge, but with the characteristic function of $X_{T-t} - X_0$ instead of $X_T - X_t$. Aside from specifying both the dynamics and the hedging strategy in each of the models, we also consider the error from specifying the hedging strategy according to a CGMY model when the underlying asset price process follows a Heston model. This gives some further insights into robustness against model risk.

We consider the hedging performance from hedging a call option with strike 0.9 expiring in half a year. In each model, we simulate the underlying asset price path 2000 times and compare the resulting P&L from hedging in a static way and in a dynamic way, where daily rebalancing takes place. For static hedges we compare three methods: First, we consider static hedges assuming that a continuum of hedging instruments is available; here the hedge error is introduced by choosing a regularisation parameter. Second, we consider the Gaussian quadrature method, and third the optimal least squares method.

Figure 6 shows box-and-whisker plots of the resulting P&L distributions. In the Black-Scholes case, the dynamic hedging strategy performs by far best, even though the largest overall loss occurs in dynamic hedging. Even if some of the static hedging strategies tend to produce profits rather than losses, the overall variation in P&L is much larger, which implies an inferior hedging quality. For the CGMY model, each of the static hedging strategies outperforms the dynamic hedging strategy. In particular, dynamic hedging produces a large number extreme losses, with the largest loss being approximately 68% of the option premium (0.1171 in the CGMY model). The situation is different for the Heston model: here, we find that the magnitude of P&L is comparable across different hedging strategies It is readily observed that the static hedges tend to produce a profit while the dynamic strategy tends to produce losses and can produce even great losses in the order of the option premium, which is 0.1138 in the Heston model. However, the direction of the P&L should not be over-interpreted, as this may well change with changes in the calibration parameters. Finally, when applying the hedging strategy as prescribed by the CGMY model to the Heston model, we find again that static hedging performs best in that it does not produce any extreme losses. Interestingly, the hedge performance is superior when compared to the hedging strategies from the Heston model.

These findings suggest that, first, dynamic hedging performs poorly in the presence of jumps when compared to static hedging and that, second, static hedging provides some robustness

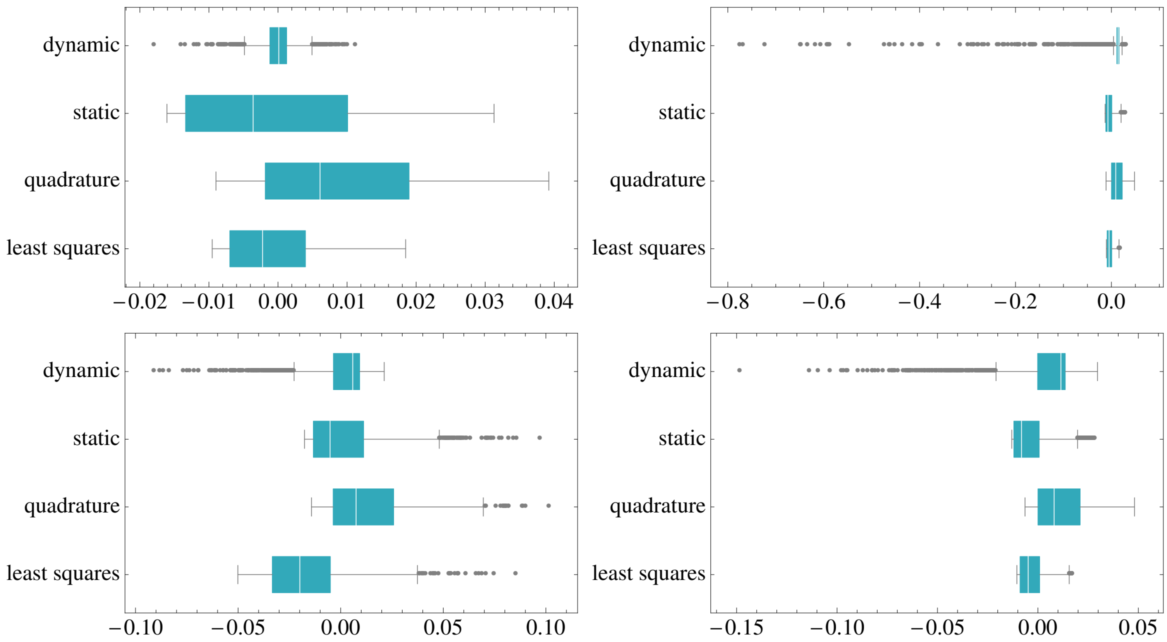

Figure 6: P&L from hedging in different ways. Top left: Black-Scholes model; top right: CGMY model; bottom left: Heston model; bottom right: CGMY hedge when underlying follows Heston model. Boxes, separated by median, denote second and third quartiles; dots denote outliers, which are outcomes that lie beyond 1.5 times the interquartile range.

against model risk.

7 Conclusion

Under fairly general conditions, we treat the problem of devising static hedges for European style payoffs maturing earlier than the hedging instruments. First, we derive existence of approximating static hedging strategies, converging almost surely and in L^1. Next, we show how to construct such hedging strategies, both when assuming a continuum of hedging instruments and when the number of hedging instruments is finite. Our results are complemented by examples. Overall, the hedging error is greatest at points where the payoff function has discontinuities or is not otherwise sufficiently smooth. When comparing static hedging with dynamic hedging, we find that in the Black-Scholes model dynamic hedging, where an error is introduced due to discrete rebalancing of the replicating portfolio, outperforms static hedging. In incomplete market models we find that static hedging outperforms dynamic hedging in the CGMY model, whereas the magnitude of P&L of different hedging strategies are of comparable size when hedging in the Heston model. This suggests that in the presence of jumps static hedging performs better than dynamic hedging.

A Application of Nyquist-Shannon Theorem

We apply the *Nyquist-Shannon-Theorem (Wittaker-Shannon Theorem, Sampling Theorem)* to derive an approximate hedging strategy for a call option.

Theorem 4 (Nyquist-Shannon Theorem)**.** *Let f be a bandlimited and square-integrable function with band interior to the interval $[-\Omega, \Omega]$. Then,*

$$f(x) = \sum_{n=-\infty}^{\infty} f\left(n\frac{\pi}{\Omega}\right) \operatorname{sinc}\left[\frac{\Omega}{\pi}\left(x - n\frac{\pi}{\Omega}\right)\right], \tag{28}$$

where $\operatorname{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$.

In the following we set $\tau := \pi/\Omega$.

Let us assume first that $\tilde{p}$ satisfies the conditions of Lemma 1. Then, assuming that $\tilde{p}$ has bandwidth Ω, and using the decomposition (28), we obtain, using that the FT is a linear transformation,

$$\begin{aligned}
\tilde{g} = \mathfrak{F}^{-1}\left(\frac{\mathfrak{F}(\tilde{p})}{\mathfrak{F}(\gamma)}\right) &= \mathfrak{F}^{-1}\left(\frac{\mathfrak{F}(\sum_{n=-\infty}^{\infty} \tilde{p}(n\tau)\operatorname{sinc}(1/\tau(x-n\tau)))}{\mathfrak{F}(\gamma)}\right) \\
&= \sum_{n=-\infty}^{\infty} \mathfrak{F}^{-1}\left(\frac{\tilde{p}(n\tau)\mathfrak{F}(\operatorname{sinc}(1/\tau(x-n\tau)))}{\mathfrak{F}(\gamma)}\right) \\
&= \sum_{n=-\infty}^{\infty} \tilde{p}(n\tau)\mathfrak{F}^{-1}\left(\frac{\mathfrak{F}(\operatorname{sinc}(1/\tau(x-n\tau)))}{\mathfrak{F}(\gamma)}\right).
\end{aligned}$$

The FT of the sinc-function is given by

$$\mathfrak{F}(\operatorname{sinc})(\omega) = \mathbf{1}_{\{-\pi,\pi\}}(\omega).$$

Using the following properties of the FT,

$$\begin{aligned}
\mathfrak{F}(f(x-x_0))(\omega) &= \mathrm{e}^{i\omega x_0}\mathfrak{F}(f)(\omega), \quad x_0 \in \mathbb{R} \\
\mathfrak{F}(f(ax))(\omega) &= \frac{1}{a}\mathfrak{F}(f)\left(\frac{\omega}{a}\right), \quad a \neq 0,
\end{aligned}$$

we obtain

$$\mathfrak{F}(\operatorname{sinc}(1/\tau(x-n\tau)))(\omega) = \tau\mathfrak{F}(\operatorname{sinc}(x-n\tau))(\omega\tau) = \tau \mathrm{e}^{i\omega \mathrm{n}\tau}\mathfrak{F}(\operatorname{sinc})(\omega\tau) = \tau\mathrm{e}^{i\omega n\tau}\mathbf{1}_{\{-\pi/\tau,\pi/\tau\}}(\omega).$$

Continuing above, we have

$$\tilde{g}(x) = \sum_{n=-\infty}^{\infty} \tilde{p}(n\tau)\mathfrak{F}^{-1}\left(\frac{\tau\mathrm{e}^{-i\omega n\tau \mathbf{1}_{\{-\pi/\tau,\pi/\tau\}}(\omega)}}{\mathfrak{F}(\gamma)(\omega)}\right)(x) = \sum_{n=-\infty}^{\infty} \tilde{p}(n\tau)\frac{1}{2\pi}\int_{-\pi/\tau}^{\pi/\tau} \frac{\tau\mathrm{e}^{i\omega n\tau}}{\mathfrak{F}(\gamma)(\omega)}\mathrm{e}^{-i\omega x}\,\mathrm{d}w,$$

which can be calculated numerically, using Equations (6) and (7) to calculate $\mathfrak{F}(\gamma)$.

If the payoff function $\tilde{p}$ is not bandlimited, one can approximate the hedging strategy by forcing $\tilde{p}$ to be bandlimited through an appropriate choice of Ω.

The example in Figure 1 is parameterised as follows: the payoff to be hedged, resp. replicated, is a call option with strike $K = 0.6$ maturing at 0.5, that is, the payoff is $p(x) = (\mathrm{e}^x - 0.6)^+$, and $\tilde{p}(x) = \mathrm{e}^{-(1+\alpha)x}(\mathrm{e}^x - 0.6)^+$. We choose $\alpha = 1$. The hedging / replicating instruments are call options maturing at time 1. We assume a Black-Scholes model for the underlying asset price process, with volatility $\sigma = 0.2$.

In order to apply the Nyquist-Shannon Theorem, we choose $\Omega = 8$, so that $\tau = \pi/4$ and furthermore, we let the sum in Equation (28) range from -20 to 20.

B Market data

Term	Expiry	Strike										
		80%	85%	90%	95%	97.50%	100%	102.50%	105%	110%	115%	120%
		1327.02	1409.96	1492.9	1575.84	1617.31	1658.78	1700.25	1741.72	1824.66	1907.6	1990.54
1 month	06/14/2013	18.265	18.265	18.265	14.812	12.929	11.439	10.719	10.918	12.721	12.794	12.794
2 months	07/14/2013	17.034	17.014	17.475	14.898	13.424	12.206	11.401	11.103	11.643	11.397	11.445
3 months	08/13/2013	20.711	20.355	17.873	15.328	14.141	13.123	12.341	11.859	12.001	13.357	13.323
6 months	11/11/2013	22.122	19.878	17.878	16.047	15.205	14.445	13.787	13.259	12.661	12.69	13.337
1 year	05/10/2014	21.609	20.083	18.636	17.288	16.664	16.075	15.526	15.033	14.219	13.67	13.417
1.5 years	11/06/2014	21.524	20.285	19.117	18.03	17.522	17.042	16.595	16.18	15.457	14.895	14.5
2 years	05/05/2015	21.699	20.646	19.656	18.735	18.305	17.897	17.511	17.147	16.499	15.953	15.512

Table 4: Implied volatilities of options on the S&P 500 index on 15 May 2013. Source: Bloomberg.

Term	Expiry	Strike										
		80%	85%	90%	95%	97.50%	100%	102.50%	105%	110%	115%	120%
1 month	06/14/2013	0.2000	0.1500	0.1004	0.0523	0.0305	0.0132	0.0038	0.0008	0.0001	0.0000	0.0000
2 months	07/14/2013	0.2000	0.1502	0.1021	0.0566	0.0364	0.0199	0.0089	0.0033	0.0004	0.0000	0.0000
3 months	08/13/2013	0.2005	0.1522	0.1050	0.0612	0.0421	0.0262	0.0144	0.0070	0.0015	0.0005	0.0001
6 months	11/11/2013	0.2048	0.1579	0.1136	0.0735	0.0560	0.0407	0.0281	0.0184	0.0069	0.0025	0.0010
1 year	05/10/2014	0.2151	0.1718	0.1315	0.0951	0.0788	0.0641	0.0509	0.0396	0.0224	0.0117	0.0059
1.5 years	11/06/2014	0.2260	0.1852	0.1473	0.1130	0.0975	0.0831	0.0701	0.0583	0.0390	0.0249	0.0154
2 years	05/05/2015	0.2372	0.1984	0.1624	0.1297	0.1147	0.1007	0.0878	0.0759	0.0555	0.0394	0.0272

Table 5: Call prices derived from implied volatilities, assuming $S_0 = 1$ and $r = q = 0$.

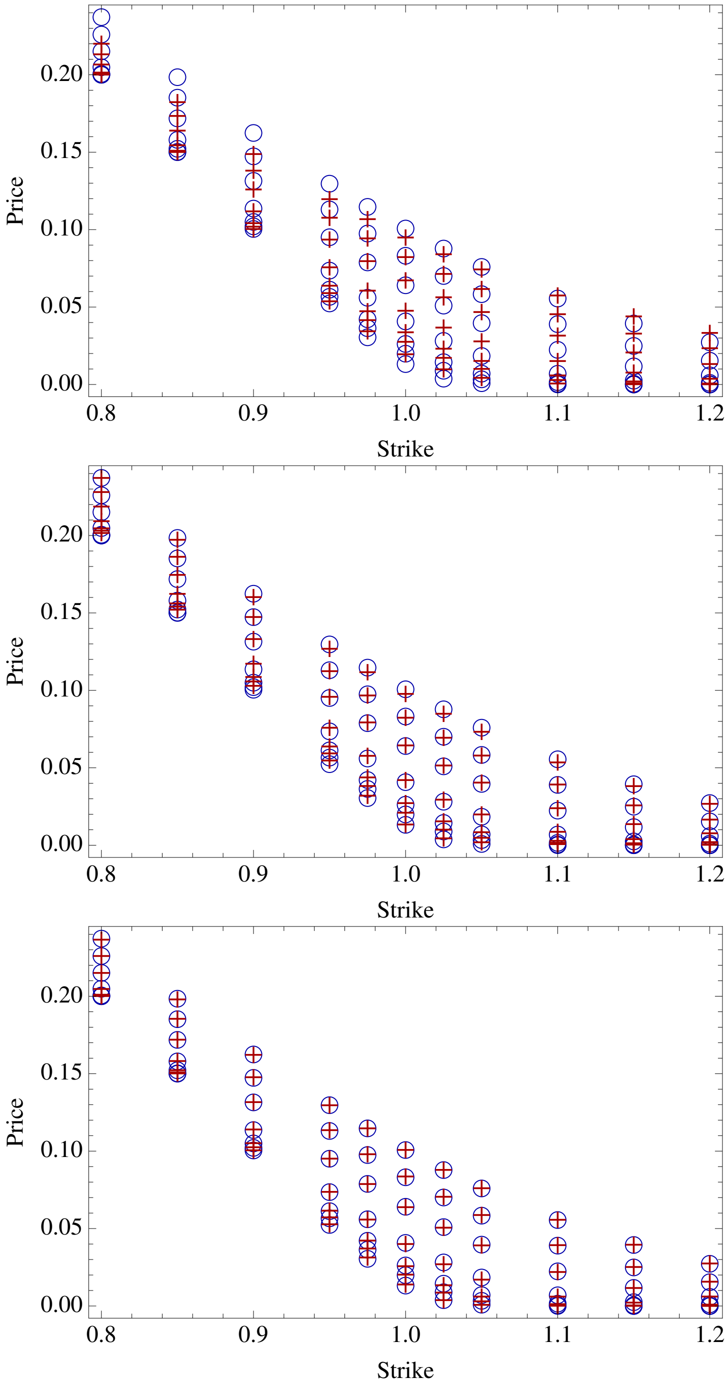

Figure 7: Calibration results. Circles are market prices of call options for several maturities and strikes, crosses are corresponding model prices. Top: Black-Scholes, middle: CGMY, bottom: Heston.

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