CORRELATION UNDER STRESS IN NORMAL VARIANCE MIXTURE MODELS

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December 23, 2012

Abstract

We investigate correlations of asset returns in stress scenarios where a common risk factor is truncated. Our analysis is performed in the class of normal variance mixture (NVM) models, which encompasses many distributions commonly used in financial modelling. For the special cases of jointly normally and t-distributed asset returns we derive closed formulas for the correlation under stress. For the NVM distribution, we calculate the asymptotic limit of the correlation under stress, which depends on whether the variables are in the maximum domain of attraction of the Fréchet or Gumbel distribution. It turns out that correlations in heavy-tailed NVM models are less sensitive to stress than in medium- or light-tailed models. Our analysis sheds light on the suitability of this model class to serve as a quantitative framework for stress testing, and as such provides valuable information for risk and capital management in financial institutions, where NVM models are frequently used for assessing capital adequacy. We also demonstrate how our results can be applied for more prudent stress testing.

Key Words: stress testing, risk management, correlation, normal variance mixture distribution, multivariate normal distribution, multivariate t-distribution

1 Introduction

In risk management, stress testing encompasses various techniques used by financial firms to test bank capital adequacy in a strongly adverse market environment. Stress tests on bank portfolios are mandatory under the Basel II regulatory rules, and have gained importance as an integral part of risk management and banking supervision in the aftermath of the subprime crisis, see e.g. Turner (2009), Larosière et al. (2009), Brunnermeier et al. (2009), BIS (2009). For example, bank supervisors conducted mandatory stress tests in the previous years to assess the adequacy of capital buffers of the largest U.S. and European banks.

A standard technique of conducting stress tests is to calculate portfolio risk measures, such as expected loss, value-at-risk, economic capital or regulatory capital under the assumption of

*The views expressed in this paper are those of the authors and do not necessarily reflect the position of Deutsche Bank AG.

†The authors wish to thank Steffi Höse, Claudia Klüppelberg, Ludger Overbeck, Ursula Walther and two anonymous referees for comments and discussions.

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adverse market conditions. Typically, stress scenarios are translated into constraints on risk factors, which are economic or market variables that affect portfolio losses. In our setup, the constraints are formalised by truncating the risk factor variables, that is, by conditioning on the range of values that a risk factor may attain. For example, a stress scenario for an equity portfolio may be defined as a decrease of an equity index (considered as a risk factor) by more than 10% within a time period of one year.

The truncation of risk factors is a commonly used stress testing technique for the credit risk management and capital management of financial institutions, see e.g. Bonti et al. (2006), Kalkbrener and Overbeck (2008), Duellmann and Erdelmeier (2009). In these papers, stress scenarios are implemented in structural credit portfolio models, which link the default of a firm to the relationship between its assets and liabilities. The dependence between assets is specified by systematic factors, which usually represent geographic regions or industries. Each stress scenario specifies a constraint on one or more systematic factors and thus quantifies the impact of a downturn in an economy or industry on the credit portfolio of a bank.[1]

The outcome of a stress test depends strongly on the correlations among portfolio components in the stress scenario. In general, the *conditional correlation* between asset returns – conditional on the truncated factor – is different from the unconditional correlation. However, the conditional correlations are the correlations that implicitly enter the calculation of risk measures under the stress scenario, since the risk measure is now calculated under a new probability measure obtained by conditioning. An analysis of conditional correlation may therefore contribute to understanding the behaviour of portfolio models under stress and to refine the stress testing methodology.

In this paper we analyse correlation under stress for a quite general class of distributions that encompasses many models commonly encountered in the financial industry. We provide closed formulas for the conditional correlation in the cases of jointly normally distributed asset returns and jointly t-distributed asset returns, which are standard distributions in portfolio modelling.[2] More generally, we consider normal variance mixture (NVM) distributions, a family of distributions with diverse tail behaviour, ranging from light- to heavy-tailed. Essentially, a random vector follows an NVM distribution if it has a representation as a jointly normally distributed random vector multiplied with a strictly positive and independent "mixing" random variable.[3] NVM distributions belong to the class of elliptical distributions, whose key parameter for describing the dependence is the correlation matrix. These properties make NVM distributions attractive for high-dimensional applications, such as modelling the asset returns of a bank portfolio. For NVM models, we calculate the asymptotic limit of the conditional correlation. Here, we employ results and techniques from extreme value theory.

It turns out that the limit of the conditional correlation depends on whether the variables in the NVM model are in the maximum domain of attraction (MDA) of a Fréchet distribution or the Gumbel distribution.[4] More precisely, consider asset returns A_i and A_j with (unconditional) correlation ρ_{ij} and a risk factor V, whose correlation with each of the asset returns is denoted

[1]For further stress test methods, see also Kupiec (1998), Berkowitz (2000) and Breuer et al. (2009); for applications of extreme value theory to stress testing, see e.g. Longin (2000) and Schachter (2001).

[2]For example, in structural credit portfolio models, such as CreditMetrics$^{\text{TM}}$, Gupton et al. (1997), and Moody's KMV Portfolio Manager$^{\text{TM}}$, Crosbie and Bohn (2002), the multivariate normal distribution is still the de-facto standard for modelling log-returns of asset values.

[3]For example, the increments of certain time-changed Brownian motion models, such as the variance gamma or the inverse Gaussian processes and certain stochastic volatility models follow an NVM distribution, see e.g. Madan and Seneta (1990), Barndorff-Nielsen (1998), Geman et al. (2001).

[4]Essentially, the MDA of the Fréchet distribution consists of the heavy-tailed distributions, such as the t-distribution, while the MDA of the Gumbel distribution contains mostly light- to medium-tailed distributions, for example, the normal and the lognormal distributions.

by ρ_i and ρ_j, respectively. We show that the asymptotic limit of the conditional correlation of A_i and A_j equals

$$\text{(1.1)} \qquad \frac{\rho_i\,\rho_j + (\rho_{ij} - \rho_i\,\rho_j)\,(\alpha - 1)}{\sqrt{(\rho_i^2 + (1-\rho_i^2)\,(\alpha-1))\,(\rho_j^2 + (1-\rho_j^2)\,(\alpha-1))}}, \quad \alpha > 2,$$

if the risk factor is stressed, i.e., if V is truncated at a threshold C, and C converges to $-\infty$. The parameter α specifies the tail index of the asset returns and the risk factor in the Fréchet case. The case when the variables are in the MDA of the Gumbel distribution corresponds to the limit as $\alpha \to \infty$, in which case the asymptotic limit of the conditional correlation between A_i and A_j is

$$\text{(1.2)} \qquad \frac{\rho_{ij} - \rho_i\,\rho_j}{\sqrt{(1-\rho_i^2)(1-\rho_j^2)}}.$$

The limit formula (1.1) of the conditional asset correlation has a number of interesting consequences. First of all, it shows that for small α the correlations in the NVM model are insensitive to stress, i.e., the conditional correlations do not differ significantly from the unconditional correlations in heavy-tailed NVM models. In the Gumbel case, on the other hand, the impact of stress on the conditional correlation is typically much stronger than for heavy-tailed models, in particular when comparing to the heavy-tailed case with α sufficiently close to 2. The asymptotic conditional correlation may be either greater or smaller than the unconditional asset correlation depending largely on the correlations between the risk factor and the respective asset returns. In particular, when the assets in question are sufficiently correlated with the risk factor, the conditional correlation is typically smaller than the unconditional correlation. Loosely speaking, in many cases systematic risk is reduced by conditioning on the risk factor, whereas unsystematic risk remains (a detailed example is given in Section 5).

The implications of our results are three-fold: first, they shed light on the impact of stress tests on the correlation of portfolio constituents and associated portfolio diversification within a model family that includes many commonly used light- and heavy-tailed distributions. Second, depending on how the conditional correlation obtained from the model reconciles with empirical observation, one may assess whether the model class is an appropriate model for the dependence structure. Furthermore, one may incorporate correlation as an additional risk factor and apply appropriate changes to the correlation itself when applying a stress scenario. We discuss these applications in our examples.

In a wider sense, our results may be of interest in the context of systemic risk measures, such as CoVaR and ΔCoVaR, suggested by Adrian and Brunnermeier (2011), and marginal expected shortfall (MES), suggested by Acharya et al. (2010). Essentially, these measures gauge risk conditioned on certain stress events, such as a bank or the system being in distress. The (model) effect of such stress events on the correlation between institutions may give additional useful information when using these measures.

The model behaviour of conditional correlation has also been studied for example by Boyer et al. (1999), where formulas for conditional asset correlations are determined for bivariate normally distributed asset returns, when one of two asset returns is stressed. Malevergne and Sornette (2006) consider in addition bivariate t-distributed asset returns. These are special cases of our analysis of the stressed NVM model, in which the stressed risk factor coincides with one of the assets. In a stress test setup similar to ours, that is, by truncating a common risk factor, Bae and Iscoe (2010) consider a jump-diffusion model in order to achieve an increasing conditional correlation.

The remainder of the paper is structured as follows: in Section 2 we introduce the model setup involving asset returns and a single risk factor. We derive a general formula for conditional asset correlations in NVM models, which is based on some conditional expectations, in Section 3. In addition, we provide explicit formulas for the cases when the risk factor and the asset returns are normally distributed and t-distributed. In Section 4, we investigate the asymptotic limit of conditional correlation as the truncation of the risk factor tends to $-\infty$. More precisely, we derive formula (1.1) for the case when the NVM distribution is in the MDA of the Fréchet distribution (under some additional moderate assumption) and formula (1.2) when it is in the MDA of the Gumbel distribution. In Section 5 we analyse the relationship between correlations under stress and Value-at-risk in structural credit portfolios. We also provide examples where the correlation itself is considered to be a risk factor and subjected to change under stress. An empirical example involving DAX data is given. Finally, we conclude and discuss the results in Section 6. Appendix A provides a brief overview of the concepts from extreme value theory that are applied in this paper.

2 Definitions and problem

Let $V, A_1, \dots, A_k$ be random variables on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$. In our setting, V will be interpreted as a risk factor, and $A_1, \dots, A_k$ may be interpreted as additional risk factors or as variables that specify asset returns or the credit quality of k assets.

We assume that $V, A_1, \dots, A_k$ follow a multivariate *normal variance mixture (NVM) distribution*, i.e., there exist normally distributed variables $X, Y_1, \dots, Y_k$ and a $\mathbf{P}$–a.s. strictly positive random variable W, independent of $X, Y_1, \dots, Y_k$, such that

$$V := \sqrt{W}X, \qquad A_i := \sqrt{W}Y_i, \qquad i = 1, \dots, k. \tag{2.1}$$

We assume in addition that W is integrable and that X and $Y_1, \dots, Y_k$ are standardized.[5] For $i, j \in \{1, \dots, k\}$ the correlations of X, Y_i, Y_j are denoted by

$$\rho_i := \mathrm{Corr}(X, Y_i), \qquad \rho_{ij} := \mathrm{Corr}(Y_i, Y_j). \tag{2.2}$$

It is straightforward to show that $\mathrm{Corr}(V, A_i) = \rho_i$ and $\mathrm{Corr}(A_i, A_j) = \rho_{ij}$. We shall also sometimes use the following orthogonalization of Y_i on X:

$$Y_i = \rho_i X + \sqrt{1 - \rho_i^2} Z_i, \tag{2.3}$$

where Z_i is a standard normally distributed variable independent of X and W.

The NVM distribution encompasses a number of distributions that are commonly used to model asset returns. Some examples of NVM distributions are the multivariate normal distribution, in which case the mixing variable W is a constant, or the multivariate t-distributions, when W follows an inverse gamma distribution. When W follows a so-called generalised inverse Gaussian distribution, then the NVM distribution is a symmetric generalised hyperbolic distribution. More generally, NVM distributions belong to the family of elliptic distributions, for which correlation appropriately describes the dependence among the variables. For a general review of NVM distributions we refer to (McNeil et al. 2005, Section 2.3) and Bingham and Kiesel (2002).

Let us consider the situation when V is truncated by $C \in \mathbb{R}$, that is, $V \leq C$ and write

$$\mathbf{P}^C(A) = \mathbf{P}(A | V \leq C), \quad A \in \mathcal{A}, \tag{2.4}$$

[5] Working with standardized variables is not a restriction, as long as we are interested only in correlations.

for the corresponding conditional distribution. In this setting, the random variable V is interpreted as a stressed risk factor with C the level of stress applied to V.

Let us rephrase the objectives of the paper: we wish to analyse the impact of the stress on the correlation of the A_i, $i = 1, \ldots, k$. In the special cases of a multivariate normal model ($W = 1$) and a jointly t-distributed model (W inverse gamma distributed), we derive formulas for conditional asset correlations $\mathrm{Corr}^C(A_i, A_j)$, where Corr^C is the correlation coefficient under the measure $\mathbf{P}^C$ (Section 3). For the general NVM model, we derive the limit of the conditional asset correlations as $C \to -\infty$ (Section 4).

3 Correlations under stress

In this section we derive a general formula for the conditional correlation $\mathrm{Corr}^C(A_i, A_j)$. We further derive explicit formulas for the case when the variables are normally distributed (Section 3.1) and when the variables are t-distributed (Section 3.2).

Proposition 3.1. *In the NVM model, and additionally writing* $\mathbb{E}^C$ *and* Var^C *for the expectation and variance under* $\mathbf{P}^C$*, respectively, the conditional correlation of* A_i*,* A_j *is given by*

$$\mathrm{Corr}^C(A_i, A_j) = \frac{\rho_i\,\rho_j\,\frac{\mathrm{Var}^C(V)}{\mathbb{E}^C(W)} + \rho_{ij} - \rho_i\,\rho_j}{\sqrt{\left(\rho_i^2\,\frac{\mathrm{Var}^C(V)}{\mathbb{E}^C(W)} + (1-\rho_i^2)\right)\left(\rho_j^2\,\frac{\mathrm{Var}^C(V)}{\mathbb{E}^C(W)} + (1-\rho_j^2)\right)}}. \tag{3.1}$$

Observe that the conditional correlation depends only on the univariate quantity $\dfrac{\mathrm{Var}^C(V)}{\mathbb{E}^C(W)}$. In the remainder of the paper we shall thus concentrate on deriving explicit formulas for this quantity and for the limit of this quantity as $C \to -\infty$.

Proof. By the independence of Z_i and V, Z_i is a standard normal random variable under $\mathbf{P}^C$, so that, by Equation (2.3),

$$\mathbb{E}^C(Z_i\,Z_j) = \mathbb{E}(Z_i\,Z_j) = \frac{\rho_{ij} - \rho_i\,\rho_j}{\sqrt{(1-\rho_i^2)\,(1-\rho_j^2)}}. \tag{3.2}$$

Also, by Equation (2.3), we have $\mathbb{E}^C(A_i) = \rho_i\,\mathbb{E}^C(V)$, so that one obtains

$$\mathrm{Var}^C(A_i) = \rho_i^2\,\mathrm{Var}^C(V) + (1-\rho_i^2)\,\mathbb{E}^C(W)$$
$$\mathrm{Cov}^C(A_i, A_j) = \rho_i\,\rho_j\,\mathrm{Var}^C(V) + (\rho_{ij} - \rho_i\,\rho_j)\,\mathbb{E}^C(W).$$

The claim follows by dividing both the conditional covariance and the conditional variance by $\mathbb{E}^C(W)$. □

Let us illustrate the impact of truncating a risk factor on the conditional correlation: The left plot of Figure 1 shows a scatterplot of 5000 simulated samples representing asset returns of two assets, A_1 and A_2. The asset returns are jointly normally distributed, each with a standard deviation of 20%, and with a correlation of $\rho_{12} = 0.6$. Let us assume that the assets are in addition correlated with a common risk factor V, which is also normally distributed with a standard deviation of 20%, and where $\rho_1 = 0.8$ and $\rho_2 = 0.7$. The right plot shows the samples conditional on $V \leq -0.3$. It appears that the conditional samples are less correlated than the unconditional samples; in fact, their correlation is approximately 0.25. We confirm this observation analytically below.

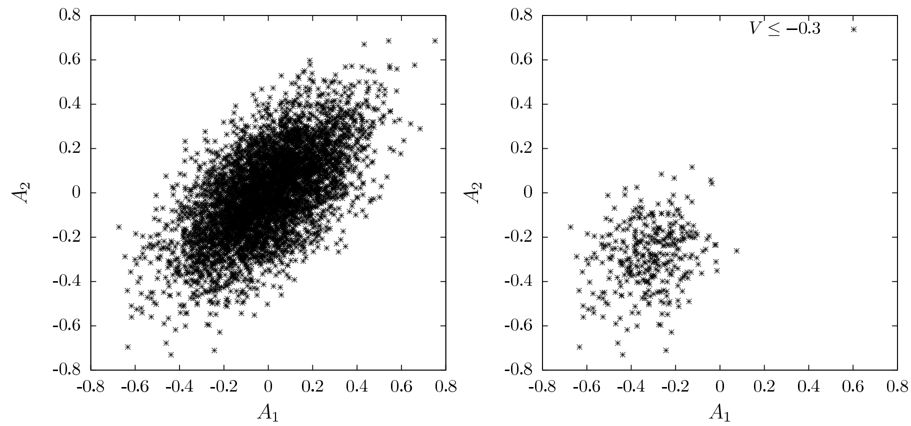

Figure 1: Left: Simulated asset returns of A_1 and A_2 that are jointly normally distributed with $\rho_{12} = 0.6$ and each with standard deviation 0.2. Right: Samples conditional on $V \le -0.3$, where V is normally distributed with standard deviation 0.2, and $\rho_1 = 0.8$ and $\rho_2 = 0.7$.

3.1 Special case: normal distribution

Let us set $W = 1$, so that $V = X$ and $A_i = Y_i$ are normally distributed. Denote by ϕ the standard normal density function and by N the standard normal distribution function.

Proposition 3.2. *Let $V, A_1, \ldots, A_k$ be standard normally distributed. Then,*

$$\operatorname{Corr}^C(A_i, A_j) = \frac{\rho_i\,\rho_j \operatorname{Var}^C(V) + \rho_{ij} - \rho_i\,\rho_j}{\sqrt{(\rho_i^2 \operatorname{Var}^C(V) + 1 - \rho_i^2)(\rho_j^2 \operatorname{Var}^C(V) + 1 - \rho_j^2)}}, \tag{3.3}$$

with

$$\operatorname{Var}^C(V) = 1 - \frac{C\,\phi(C)}{\mathrm{N}(C)} - \frac{(\phi(C))^2}{(\mathrm{N}(C))^2}. \tag{3.4}$$

Proof. The first equation follows directly from Proposition 3.1, so it remains to prove Equation (3.4). First, observe that $\mathbb{E}(V\,\mathbf{1}_{\{V \le C\}}) = -\phi(C)$, which follows directly from $\phi(C) = \int_{-\infty}^{C} \phi'(x)\,\mathrm{d}x = \int_{-\infty}^{C} -x\,\phi(x)\,\mathrm{d}x$. Second, observe that $\mathbb{E}(V^2\,\mathbf{1}_{\{V \le C\}}) = \mathrm{N}(C) - C\,\phi(C)$, which follows from $C\,\phi(C) = \int_{-\infty}^{C} (x\,\phi(x))'\,\mathrm{d}x = \mathrm{N}(C) - \int_{-\infty}^{C} x^2\,\phi(x)\,\mathrm{d}x$. The claim is now obtained via

$$\operatorname{Var}^C(V) = \mathbb{E}^C(V^2) - \big(\mathbb{E}^C(V)\big)^2 = \frac{\mathbb{E}(V^2\,\mathbf{1}_{\{V \le C\}})}{\mathbf{P}(V \le C)} - \frac{\mathbb{E}(V\,\mathbf{1}_{\{V \le C\}})^2}{\mathbf{P}(V \le C)^2}.$$

□

Let us illustrate conditional correlation for five examples. The data are given in Table 1. Here, ρ_{12} specifies the correlation between assets A_1 and A_2, and ρ_i is the correlation between the risk factor and asset A_i, $i = 1, 2$. Since all variables are standardized, the correlation $\operatorname{Corr}(Z_1, Z_2)$ of the orthogonalization of Equation (2.3) is given by Equation (3.2). In all examples, we have set $\rho_{12} = 0.6$. In example 1, the first asset coincides with the risk factor. In examples 2 and 3, both assets are quite strongly correlated with the risk factor. In example 4, both assets are only weakly correlated with the risk factor, and in example 5, A_1 is strongly correlated with the risk factor, while the correlation between A_2 and the risk factor is low. The conditional correlations as a function of the truncation level are shown in Figure 2. When

Table 1: Example parameters.

Example	ρ_{12}	ρ_1	ρ_2	$\mathrm{Corr}(Z_1, Z_2)$
1	0.6	1	0.6	0
2	0.6	0.8	0.7	0.093
3	0.6	0.6	0.6	0.375
4	0.6	0.1	0.1	0.596
5	0.6	0.7	0.02	0.82

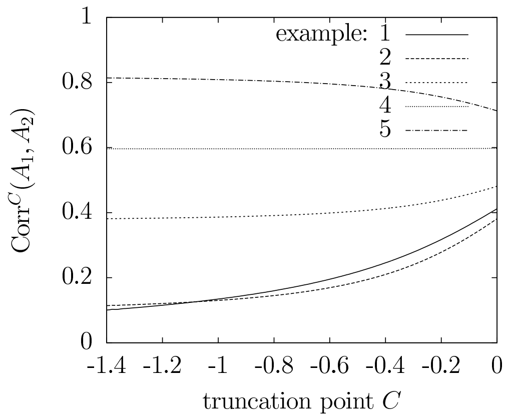

Figure 2: Conditional asset correlations when V, A_1, A_2 are normally distributed. The risk factor V has a standard deviation of 20%; for example, one may think of C as the truncation level of an equity index return with volatility 20%.

the dependence of both assets on the risk factor is sufficiently high, the conditional correlation decreases as the truncation level decreases. For example, assume that V denotes the return on an equity index and let A_1 and A_2 be the returns on two index components. Clearly, a negative return smaller than C on the equity index return V translates into a strong negative return on two index constituents A_1 and A_2. However, conditional on $V \leq C$, the correlation between A_1 and A_2 is now driven mainly by the correlation between the specific components Z_1 and Z_2, as demonstrated in Figure 1. In fact, we show in Section 4.1 that the conditional correlation converges to $\mathrm{Corr}(Z_i, Z_j)$ as $C \to -\infty$. This also explains why in Example 5 the conditional correlation increases with the stress level. For example, there could be a second risk factor driving the correlation of Z_1 and Z_2. This second risk factor becomes more dominant as the stress level increases.

3.2 Special case: t-distribution

The class of multivariate t-distributions is frequently used in the finance industry to model random variables with heavy tails. We derive an explicit formula for conditional correlations in t-distributed models. As outlined above, it is sufficient to calculate $\mathrm{Var}^C(V)/\mathbb{E}^C(W)$, which can be expressed in terms of beta functions.

Proposition 3.3. *Let $V, A_1, \ldots, A_k$ follow a multivariate t-distribution with parameter $\nu > 2$ denoting the degrees of freedom. Then, for $C < 0$,*

$$\frac{\mathrm{Var}^C(V)}{\mathbb{E}^C(W)} = \frac{\mathbb{E}^C(V^2) - \mathbb{E}^C(V)^2}{\mathbb{E}^C(W)} = \frac{f(\nu, C)}{g(\nu, C)},$$

with

$$f(\nu,C) := B\left(\frac{\nu}{C^2+\nu};\frac{\nu-2}{2},\frac{3}{2}\right) - \frac{4\left(\frac{\nu}{C^2+\nu}\right)^{\nu-1}}{(\nu-1)^2 B\left(\frac{\nu}{C^2+\nu};\frac{\nu}{2},\frac{1}{2}\right)}$$

$$g(\nu,C) := \frac{B\left(\frac{1}{2},\frac{\nu}{2}\right)}{\nu-2} - \frac{\left(B\left(\frac{\nu-2}{2},\frac{1}{2}\right) - B\left(\frac{\nu}{\nu+C^2};\frac{\nu-2}{2},\frac{1}{2}\right)\right)}{\nu-1},$$

where $B(y;a,b) := \int_0^y t^{a-1}(1-t)^{b-1}\,\mathrm{d}t$ *is the incomplete beta function and* $B(a,b) := B(1;a,b)$ *is the beta function.*

Proof.[6] For $n \in \mathbb{N}$, $n < \nu$, and for $C < 0$ we first show that the conditional moments of V are given by

$$\mathbb{E}^C(V^n) = \frac{(-1)^n\,\nu^{n/2}\,B\left(\frac{\nu}{C^2+\nu};\frac{\nu-n}{2},\frac{n+1}{2}\right)}{B\left(\frac{\nu}{C^2+\nu};\frac{\nu}{2},\frac{1}{2}\right)}. \tag{3.5}$$

Since V is t-distributed, we have

$$\mathbb{E}\left(V^n\,\mathbf{1}_{\{V\le C\}}\right) = \frac{\int_{-\infty}^C x^n\left(\frac{\nu}{x^2+\nu}\right)^{\frac{\nu+1}{2}}\,\mathrm{d}x}{\sqrt{\nu}B\left(\frac{\nu}{2},\frac{1}{2}\right)}. \tag{3.6}$$

Integration by substitution with variable transformation $x = -\sqrt{\left(\frac{1}{y}-1\right)\nu}$ yields

$$\begin{aligned}\int_{-\infty}^C x^n\left(\frac{\nu}{x^2+\nu}\right)^{\frac{\nu+1}{2}}\,\mathrm{d}x &= \frac{1}{2}(-1)^n\nu^{\frac{n+1}{2}}\int_0^{\frac{\nu}{C^2+\nu}}(1-y)^{\frac{n-1}{2}}y^{\frac{\nu-n-2}{2}}\,\mathrm{d}y\\ &= \frac{1}{2}(-1)^n\nu^{\frac{n+1}{2}}B\left(\frac{\nu}{C^2+\nu};\frac{\nu-n}{2},\frac{n+1}{2}\right).\end{aligned} \tag{3.7}$$

By combining Equations (3.6) and (3.7) we obtain

$$\mathbb{E}\left(V^n\,\mathbf{1}_{\{V\le C\}}\right) = \frac{(-1)^n\nu^{n/2}B\left(\frac{\nu}{C^2+\nu};\frac{\nu-n}{2},\frac{n+1}{2}\right)}{2B\left(\frac{\nu}{2},\frac{1}{2}\right)}.$$

Dividing by $\mathbf{P}(V \le C) = B\left(\frac{\nu}{C^2+\nu};\frac{\nu}{2},\frac{1}{2}\right) / \left(2B\left(\frac{\nu}{2},\frac{1}{2}\right)\right)$ yields Equation (3.5).

A simple representation can be obtained for the first conditional moment $\mathbb{E}^C(V)$: it is an immediate consequence of the definition of incomplete beta functions that

$$B\left(\frac{\nu}{C^2+\nu};\frac{\nu-1}{2},1\right) = \int_0^{\frac{\nu}{C^2+\nu}} t^{\frac{\nu-1}{2}-1}\,\mathrm{d}t = \frac{2\left(\frac{\nu}{C^2+\nu}\right)^{\frac{\nu-1}{2}}}{\nu-1},$$

and therefore

$$\mathbb{E}^C(V) = \frac{2\left(C^2+\nu\right)^{\frac{1-\nu}{2}}\nu^{\frac{\nu}{2}}}{(1-\nu)\,B\left(\frac{\nu}{C^2+\nu};\frac{\nu}{2},\frac{1}{2}\right)}. \tag{3.8}$$

[6]We are grateful to an anonymous referee whose suggestions significantly simplified the initial proof, which was based on hypergeometric functions.

Turning to the calculation of the conditional mean of W we now show that, for $C<0$,

$$\mathbb{E}^C(W)=\left(\frac{\nu B\left(\frac{\nu}{2},\frac{1}{2}\right)}{\nu-2}-\frac{\nu\left(B\left(\frac{\nu-2}{2},\frac{1}{2}\right)-B\left(\frac{\nu}{C^2+\nu};\frac{\nu-2}{2},\frac{1}{2}\right)\right)}{\nu-1}\right)\Big/B\left(\frac{\nu}{C^2+\nu};\frac{\nu}{2},\frac{1}{2}\right). \tag{3.9}$$

Since X and W are independent and ν/W follows a chi-square distribution with ν degrees of freedom, we have

$$\begin{aligned}\mathbb{E}\left(W\,\mathbf{1}_{\{V\le C\}}\right)&=\mathbb{E}\left(W\,\mathbf{1}_{\{X\le C/\sqrt{W}\}}\right)\\&=\int_0^\infty\int_{-\infty}^{C\sqrt{\frac{y}{\nu}}}\left(\frac{\nu}{y}\right)\left(\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\right)\left(\frac{2^{-\nu/2}e^{-y/2}y^{\frac{\nu}{2}-1}}{\Gamma\left(\frac{\nu}{2}\right)}\right)\mathrm{d}x\,\mathrm{d}y\\&=\frac{2^{(-\nu-1)/2}\nu}{\sqrt{\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\int_0^\infty\int_{-\infty}^{C\sqrt{\frac{y}{\nu}}}e^{\frac{-x^2-y}{2}}y^{\frac{\nu}{2}-2}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Integration by substitution with variable transformation $x=-\sqrt{z}$ yields

$$\int_0^\infty\int_{-\infty}^{C\sqrt{\frac{y}{\nu}}}e^{\frac{-x^2-y}{2}}y^{\frac{\nu}{2}-2}\,\mathrm{d}x\,\mathrm{d}y=\int_0^\infty\int_{\frac{C^2y}{\nu}}^\infty\frac{e^{\frac{-y-z}{2}}y^{\frac{\nu}{2}-2}}{2\sqrt{z}}\,\mathrm{d}z\,\mathrm{d}y.$$

For the function

$$F(t):=\int_0^\infty\int_{\frac{ty}{\nu}}^\infty\frac{e^{\frac{-y-z}{2}}y^{\frac{\nu}{2}-2}}{2\sqrt{z}}\,\mathrm{d}z\,\mathrm{d}y$$

we use the representation $F(t)=F(0)+\int_0^t F'(x)\,\mathrm{d}x$ and obtain

$$\int_0^\infty\int_{\frac{C^2y}{\nu}}^\infty\frac{e^{\frac{-y-z}{2}}y^{\frac{\nu}{2}-2}}{2\sqrt{z}}\,\mathrm{d}z\,\mathrm{d}y=F(C^2)=2^{\frac{\nu-3}{2}}\sqrt{\pi}\,\Gamma\left(\frac{\nu}{2}-1\right)-\frac{2^{\frac{\nu-3}{2}}\Gamma\left(\frac{\nu-1}{2}\right)}{\nu}\int_0^{C^2}\frac{\left(\frac{x}{\nu}+1\right)^{\frac{1}{2}-\frac{\nu}{2}}}{\sqrt{\frac{x}{\nu}}}\,\mathrm{d}x.$$

Integration by substitution with variable transformation $x=-\left(1-\frac{1}{z}\right)\nu$ yields

$$\int_0^{C^2}\frac{\left(\frac{x}{\nu}+1\right)^{\frac{1}{2}-\frac{\nu}{2}}}{\sqrt{\frac{x}{\nu}}}\,\mathrm{d}x=\int_{\frac{\nu}{C^2+\nu}}^1\frac{z^{\frac{\nu-4}{2}}\nu}{\sqrt{1-z}}\,\mathrm{d}z=\nu\left(B\left(\frac{\nu-2}{2},\frac{1}{2}\right)-B\left(\frac{\nu}{C^2+\nu};\frac{\nu-2}{2},\frac{1}{2}\right)\right).$$

By combining the equations above we obtain

$$\begin{aligned}&E\left(W\,\mathbf{1}_{\{V\le C\}}\right)\\&\quad=\frac{\nu}{4\sqrt{\pi}\,\Gamma\left(\frac{\nu}{2}\right)}\left(\sqrt{\pi}\,\Gamma\left(\frac{\nu}{2}-1\right)-\Gamma\left(\frac{\nu-1}{2}\right)\left(B\left(\frac{\nu-2}{2},\frac{1}{2}\right)-B\left(\frac{\nu}{C^2+\nu};\frac{\nu-2}{2},\frac{1}{2}\right)\right)\right).\end{aligned}$$

The basic identities

$$\Gamma(1/2)=\sqrt{\pi},\qquad\Gamma(a+1)=\Gamma(a)\,a,\qquad B(a,b)=\Gamma(a)\Gamma(b)/\Gamma(a+b),$$

can now be used to simplify the representation of $\mathbb{E}\left(W\,\mathbf{1}_{\{V\le C\}}\right)$:

$$\mathbb{E}\left(W\,\mathbf{1}_{\{V\le C\}}\right)=\frac{\nu}{2\nu-4}-\frac{\nu\left(B\left(\frac{\nu-2}{2},\frac{1}{2}\right)-B\left(\frac{\nu}{\nu+C^2};\frac{\nu-2}{2},\frac{1}{2}\right)\right)}{2(\nu-1)B\left(\frac{\nu}{2},\frac{1}{2}\right)}.$$

Dividing by $\mathbf{P}(V\le C)=B\left(\frac{\nu}{C^2+\nu};\frac{\nu}{2},\frac{1}{2}\right)/\left(2B\left(\frac{\nu}{2},\frac{1}{2}\right)\right)$ yields Equation (3.9).

The representation $\mathrm{Var}^C(V)/\mathbb{E}^C(W)=f(\nu,C)/g(\nu,C)$ is now an immediate consequence of Equations (3.5), (3.8) and (3.9). □

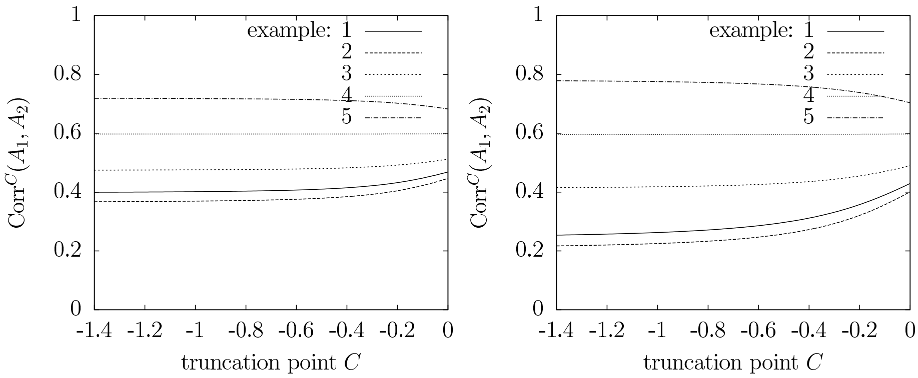

Figure 3: Conditional asset correlations when V, A_1, A_2 are t-distributed with parameter $\nu = 4$ (left) and $\nu = 10$ (right). The risk factor V has a standard deviation of 20%.

Examples of the conditional asset correlation for t-distributed models with parameter $\nu = 4$ and $\nu = 10$ are given in Figure 3. As before the examples refer to the data in Table 1. As in the normally distributed case, the conditional correlations increase or decrease depending on the correlations with the risk factor V. In the more heavy-tailed case ($\nu = 4$), the effect is much less pronounced than in the less heavy-tailed case ($\nu = 10$), which is in turn less pronounced than the normally distributed case. In other words, correlations in t-distributed models with rather heavy tails are quite insensitive to stress, whereas correlations in normally distributed models are much more affected by stress.

4 Asymptotic limit of correlations under stress

In this section we derive asymptotic limits for conditional correlations as the truncation level C tends to $-\infty$. Using Proposition 3.1, $\lim_{C\to-\infty} \mathrm{Corr}^C(A_i, A_j)$ is determined by

$$\lim_{C\to-\infty} \frac{\mathrm{Var}^C(V)}{\mathbb{E}^C(W)}, \tag{4.1}$$

if this limit exists. For normally distributed V the limit (4.1) can be computed using only properties of the normal distribution; see Section 4.1.

More generally, it turns out that essentially the limit exists whenever V is in the maximum domain of attraction (MDA) of some extreme value (EV) distribution.[7] A random variable X is in the MDA of the distribution H if H is non-degenerate and if a scaled and normalized distribution of the maximum M_n of n iid copies of X converges to H. The limit distribution H is one of the Fréchet, Gumbel or Weibull EV distributions. We write $X \in \mathrm{MDA}(H)$ (or $F \in \mathrm{MDA}(H)$, where F is a distribution function) to mean that X (F) is in the MDA of H, cf. Appendix A.

The limit (4.1) depends on whether W is in the MDA of the Fréchet distribution, in which case V is in the Fréchet MDA, or whether V is in the MDA of the Gumbel distribution.[8] We treat the two cases separately in Sections 4.2 and 4.3, respectively.

The following result, required later, holds for all NVM-distributed variables.

[7] Basically all continuous distributions commonly used in statistics are in the MDA of some EV distribution, see (McNeil et al. 2005, Section 7.1).

[8] The case that V is in the MDA of the Weibull distribution does not arise, as all distributions in this MDA have a finite right endpoint.

Lemma 4.1. *On $\mathbb{R} \setminus \{0\}$, V has a density, given by* $f_V(x) = \mathbb{E}\left(\frac{1}{\sqrt{W}}\,\phi\left(\frac{x}{\sqrt{W}}\right)\right)$.

Proof. We have

$$F_V(x) = \mathbf{P}\left(\sqrt{W}\,X \le x\right) = \mathbb{E}\left(\mathbf{P}\left(X \le \frac{x}{\sqrt{W}}\Big| W\right)\right) = \mathbb{E}\left(\mathrm{N}\left(\frac{x}{\sqrt{W}}\right)\right).$$

For every $x \neq 0$, the family of derivatives of $\mathrm{N}(x/\sqrt{w})$, $w > 0$, is locally bounded (that is, there exist a neighbourhood U of x and a constant M such that for all $u \in U$ and for all w the derivatives are bounded by M); this follows from the continuity of the derivatives in x and w and by observing that the limits tend to zero as $w \to 0$ and as $w \to \infty$. The claim now follows by the Fundamental Theorem of Calculus and by Dominated Convergence. □

4.1 Special case: normal distribution

Let us first analyse the case of jointly normally distributed V and $A_1, \ldots, A_k$ (that is, $W = 1$). Although this is a special case of V in the MDA of the Gumbel distribution, we examine this special case separately as it requires only properties of the normal distribution, whereas the more general result relies on techniques from extreme value theory.

Proposition 4.2. *Let V be a standard normally distributed random variable. Then,*

$$\lim_{C \to -\infty} \mathrm{Var}^C(V) = 0.$$

By inserting this into the formula for asset correlations, Equation (3.1), resp. Equation (3.3), one immediately obtains the limit of the conditional correlation:

Corollary 4.3. *Let $V, A_1, \ldots, A_k$ be jointly normally distributed. Then, for $|\rho_i| < 1$ and $|\rho_j| < 1$,*

$$\lim_{C \to -\infty} \mathrm{Corr}^C(A_i, A_j) = \frac{\rho_{ij} - \rho_i\,\rho_j}{\sqrt{(1-\rho_i^2)(1-\rho_j^2)}} = \mathrm{Corr}(Z_i, Z_j). \tag{4.2}$$

If either $|\rho_i| = 1$ or $|\rho_j| = 1$ (but not both), then $\lim_{C\to-\infty} \mathrm{Corr}^C(A_i, A_j) = 0$. *If $\rho_i = \rho_j = \pm 1$, then* $\lim_{C\to-\infty} \mathrm{Corr}^C(A_i, A_j) = 1$. *If $\rho_i = -\rho_j = \pm 1$, then* $\lim_{C\to-\infty} \mathrm{Corr}^C(A_i, A_j) = -1$.

According to Equation (4.2), the correlation under stress of A_i, A_j converges towards the correlation of the residuals Z_i and Z_j obtained by regressing Y_i and Y_j on X.

Proof of Proposition 4.2. The claim follows from Equation (3.4), if we show that[9]

$$\lim_{C \to -\infty} \left[\frac{C\phi(C)}{\mathrm{N}(C)} + \frac{\phi(C)^2}{\mathrm{N}(C)^2}\right] = 1. \tag{4.3}$$

Let $C < 0$. We shall use the identity $\mathrm{N}(C) = \phi(C)\,f(C)$, for $C < 0$, where f is the continued fraction

$$f(C) = \left[\cfrac{1}{-C + \cfrac{1}{-C + \cfrac{2}{-C + \cfrac{3}{-C + \cdots}}}}\right],$$

[9] It is known from (Sampford 1953) that for $C < 0$ the expression in brackets is in $(0, 1)$ and strictly decreasing.

see (Abramowitz and Stegun 1972, Equation 26.2.14). We obtain

$$\begin{aligned}\frac{C\phi(C)}{\mathrm{N}(C)} + \frac{\phi(C)^2}{\mathrm{N}(C)^2} &= \frac{C}{f(C)} + \frac{1}{f(C)^2} = \frac{C}{\frac{1}{-C+\frac{1}{-C+\frac{2}{-C+\cdots}}}} + \left[\frac{1}{\frac{1}{-C+\frac{1}{-C+\frac{2}{-C+\cdots}}}}\right]^2 \\ &= C\left[-C + \frac{1}{-C+\frac{2}{-C+\cdots}}\right] + \left[-C + \frac{1}{-C+\frac{2}{-C+\cdots}}\right]^2 \\ &= \left[-C + \frac{1}{-C+\frac{2}{-C+\cdots}}\right] \cdot \left[C - C + \frac{1}{-C+\frac{2}{-C+\cdots}}\right] \\ &= \frac{-C}{-C+\frac{2}{-C+\cdots}} + \left[\frac{1}{-C+\frac{2}{-C+\cdots}}\right]^2 = 1 - \frac{\frac{2}{-C+\cdots}}{-C+\frac{2}{-C+\cdots}} + \left[\frac{1}{-C+\frac{2}{-C+\cdots}}\right]^2.\end{aligned}$$

Observing that $\frac{1}{-C+\frac{2}{-C+\cdots}} \leq -\frac{1}{C}$, and $\frac{2}{-C+\cdots} \leq -\frac{2}{C}$, for $C < 0$, Equation (4.3) is obtained by taking the limit. □

Figure 4 shows the asymptotic asset correlation for the examples of Table 1, where the normally distributed case corresponds to the figure labeled "(Gumbel)".

4.2 W and V in the Fréchet MDA

Let us assume that $W \in \mathrm{MDA}(\Phi_{\alpha/2})$ (i.e., W is in the Fréchet MDA with tail index $\alpha/2$), with $\alpha > 2$. Lemma 4.4 below establishes that $V \in \mathrm{MDA}(\Phi_\alpha)$. Hence, the tail distribution function $\bar{F}_V(x) = 1 - F_V(x)$ is regularly varying (at ∞) with index $-\alpha$ and can be represented as $\bar{F}_V(x) = x^{-\alpha} L(x)$, for some slowly varying function L. Some members of the Fréchet MDA are the t-distribution, the Pareto distribution and the inverse gamma distribution.

Lemma 4.4. *If W is in MDA($\Phi_{\alpha/2}$), then $\sqrt{W}$ and V are in MDA(Φ_α). Furthermore,*

$$\lim_{x\to\infty} \frac{\bar{F}_V(x)}{\bar{F}_{\sqrt{W}}(x)} = \mathbb{E}\left(X^\alpha \mathbf{1}_{\{X\geq 0\}}\right).$$

Proof. By Theorem A.4 in the Appendix, $\bar{F}_W$ can be written as $\bar{F}_W(x) = x^{-\alpha/2}L(x)$, for $x > 0$ and some slowly-varying function L. From $\bar{F}_{\sqrt{W}}(x) = \bar{F}_W(x^2) = x^{-\alpha}L(x^2)$, we obtain that $\sqrt{W} \in \mathrm{MDA}(\Phi_\alpha)$. The claim then follows directly from (Breiman 1965, Proposition 3 and Equation (3.1)) or (McNeil et al. 2005, Theorem 7.35). □

The main result of this section is the following proposition.

Proposition 4.5. *Let $W \in MDA(\Phi_{\alpha/2})$, $\alpha > 2$. Then,*

$$\lim_{C\to-\infty} \frac{\mathrm{Var}^C(V)}{\mathbb{E}^C(W)} = \frac{1}{\alpha - 1}.$$

By inserting the previous result into Equation (3.1) we obtain the asymptotic limit of $\mathrm{Corr}^C(A_i, A_j)$ for heavy-tailed factors:

Corollary 4.6. *Let $W \in MDA(\Phi_{\alpha/2})$, with $\alpha > 2$. Then*

$$\lim_{C\to-\infty} \operatorname{Corr}^C(A_i, A_j) = \frac{\rho_i\,\rho_j + (\rho_{ij} - \rho_i\,\rho_j)\,(\alpha-1)}{\sqrt{(\rho_i^2 + (1-\rho_i^2)\,(\alpha-1))\,(\rho_j^2 + (1-\rho_j^2)\,(\alpha-1))}}.$$

If V is t-distributed with parameter ν (i.e., W follows an inverse gamma distribution with parameters $\nu/2$, $\nu/2$), then the formula above holds with $\alpha = \nu$. Observe that as α approaches the value 2, the limit correlation tends to ρ_{ij}, which is just the unconditional correlation. In other words, when the tail is very heavy, then correlations are unaffected by stress testing.

Figure 4 shows the asymptotic correlation (for the heavy-tailed case as derived in Corollary 4.6) as a function of the tail index α for the five examples of Table 1. This illustrates once again that the sensitivity of conditional correlation to stress increases with the tail index α. In other words, the greater α, the greater the deviation from the unconditional correlation.

For the proof of Proposition 4.5 we need the following auxiliary result first.

Lemma 4.7. *Let X be a non-negative random variable with distribution function F and with $\bar{F}(x) > 0$. If $\mathbb{E}(X^k) < \infty$, $k \geq 0$, then*

$$\mathbb{E}(X^k|X > x) = x^k + k\int_x^\infty \frac{\bar{F}(y)\,y^{k-1}}{\bar{F}(x)}\,\mathrm{d}y, \qquad x \geq 0.$$

Moreover, if $X \in MDA(\Phi_\alpha)$, then, for $k < \alpha$, $\mathbb{E}(X^k) < \infty$ and

$$\lim_{x\to\infty} \frac{\mathbb{E}(X^k|X > x)}{x^k} = \frac{\alpha}{\alpha - k}.$$

Proof. We have $\mathbb{E}(X^k|X > x) = \mathbb{E}(X^k\,\mathbf{1}_{\{X>x\}})/\bar{F}(x)$. For the numerator we obtain that

$$\mathbb{E}(X^k\,\mathbf{1}_{\{X>x\}}) = x^k\,\bar{F}(x) + \int_x^\infty k\,t^{k-1}\,\bar{F}(t)\,\mathrm{d}t,$$

e.g. by (Kallenberg 2001, Lemma 3.4), and the first claim follows.

For the second claim, observe first that $\mathbb{E}(X^k) < \infty$ if $k < \alpha$, see (Embrechts et al. 1997, Proposition A 3.8). Let L be the slowly varying function defined by $\bar{F}(x) = x^{-\alpha}L(x)$. Applying Karamata's Theorem (Theorem A.5), yields

$$\lim_{x\to\infty} \frac{\mathbb{E}(X^k|X > x)}{x^k} = \lim_{x\to\infty} \frac{k\int_x^\infty y^{k-1-\alpha}L(y)\,\mathrm{d}y}{x^{k-\alpha}\,L(x)} + 1 = \frac{\alpha}{\alpha - k}.$$

□

Proof of Proposition 4.5. First, let us show that

$$\lim_{C\to-\infty} \frac{\operatorname{Var}^C(V)}{C^2} = \frac{\alpha}{(\alpha-2)(\alpha-1)^2}. \tag{4.4}$$

Note that $V = \sqrt{W}X$ is symmetric about 0, that is, $F_V(x) = \bar{F}_V(-x)$ for all $x \in \mathbb{R}$. Hence,

$$\begin{aligned}\lim_{C\to-\infty} \frac{\operatorname{Var}^C(V)}{C^2} &= \lim_{C\to-\infty} \frac{\mathbb{E}(V^2|V \leq C) - [\mathbb{E}(V|V \leq C)]^2}{C^2} \\ &= \lim_{C\to\infty} \frac{\mathbb{E}(V^2|V \geq C) - [\mathbb{E}(V|V \geq C)]^2}{C^2} \\ &= \frac{\alpha}{\alpha-2} - \left(\frac{\alpha}{\alpha-1}\right)^2,\end{aligned}$$

where the last step follows from Lemma 4.7, and Equation (4.4) follows.

Second, let us show that

$$\lim_{C\to-\infty}\frac{\mathbb{E}^C(W)}{C^2}=\frac{\alpha}{(\alpha-2)(\alpha-1)}. \tag{4.5}$$

Observe that, for $C\geq 0$, $\mathbb{E}^{-C}(W)=\mathbb{E}(W|\sqrt{W}X\geq C)$. Now choose a nonnegative sequence $(c_n)_{n\geq 1}$ such that $c_n\to\infty$ as $n\to\infty$, and write

$$\lim_{n\to\infty}\frac{\mathbb{E}^{-c_n}(W)}{c_n^2}=\lim_{n\to\infty}\frac{\mathbb{E}(W|\sqrt{W}X\geq c_n)}{c_n^2}=\lim_{n\to\infty}\frac{\mathbb{E}(W\,\mathbf{1}_{\{\sqrt{W}X\geq c_n\}})}{c_n^2\,\bar{F}_V(c_n)}$$
$$=\lim_{n\to\infty}\mathbb{E}\left[\frac{\mathbb{E}\left(W\,\mathbf{1}_{\{\sqrt{W}X\geq c_n\}}|X\right)}{c_n^2\,\bar{F}_V(c_n)}\right]. \tag{4.6}$$

In order to exchange the order of limit and expectation, we must show that the sequence of random variables $G_n(X)$, $n\geq 1$, defined **P**–a.s. by

$$G_n(X):=\frac{\mathbb{E}\left(W\,\mathbf{1}_{\{\sqrt{W}X\geq c_n\}}|X\right)}{c_n^2\,\bar{F}_V(c_n)},\quad n\geq 1,$$

is uniformly integrable and that $G_n(X)$, $n\geq 1$, converges **P**–a.s.. To establish uniform integrability we have to show that

$$\lim_{D\to\infty}\limsup_n\int_0^\infty G_n(x)\,\mathbf{1}_{\{G_n(x)>D\}}\,\mathrm{N}(\mathrm{d}x)=0. \tag{4.7}$$

Let L be the slowly varying function that satisfies $\bar{F}_V(x)=x^{-\alpha}L(x)$, with α the tail index of V.

First, choose $\bar{C}>0$ such that for all $c>\bar{C}$ the following four conditions are satisfied:

$$\frac{\mathbb{E}(W|\sqrt{W}>c)}{c^2}\leq\frac{\alpha}{\alpha-2}+1, \tag{4.8}$$

$$\bar{F}_{\sqrt{W}}(c)\,\mathbb{E}\left((X\mathbf{1}_{\{X>0\}})^\alpha\right)\leq 2\bar{F}_V(c), \tag{4.9}$$

$$\frac{x^{\alpha-2}L(c/x)}{L(c)}<2,\qquad 0<x\leq 1, \tag{4.10}$$

$$\frac{x^{\alpha-2}L(c/x)}{L(c)}<2\,x^{\alpha-1},\qquad 1\leq x\leq c/\bar{C}. \tag{4.11}$$

The existence of $\bar{C}$ satisfying Conditions (4.8) and (4.9) follows from Lemmas 4.4 and 4.7. Conditions (4.10) and (4.11) are a consequence of Theorem A.6 with parameters $\varepsilon=1$ and $\delta=\alpha-2$ for (4.10) and $\varepsilon=1$ and $\delta=1$ for (4.11).

Now, let $c_n\geq\bar{C}$ and $x\leq c_n/\bar{C}$. Then, by Equations (4.8) and (4.9),

$$G_n(x)=\frac{\mathbb{E}\left(W\,\mathbf{1}_{\{\sqrt{W}x\geq c_n\}}\right)}{c_n^2\,\bar{F}_V(c_n)}=\frac{\int_{c_n/x}^\infty w^2\,F_{\sqrt{W}}(\mathrm{d}w)}{c_n^2\,\bar{F}_V(c_n)}$$
$$=\left(\frac{\int_{c_n/x}^\infty w^2\,F_{\sqrt{W}}(\mathrm{d}w)}{(c_n/x)^2\,\bar{F}_{\sqrt{W}}(c_n/x)}\right)\Big/\left(\frac{c_n^2\,\bar{F}_V(c_n)}{(c_n/x)^2\,\bar{F}_{\sqrt{W}}(c_n/x)}\right)$$
$$\leq\left(\frac{\alpha}{\alpha-2}+1\right)\frac{\bar{F}_{\sqrt{W}}(c_n/x)}{x^2\,\bar{F}_V(c_n)}$$
$$\leq H(\alpha)\frac{\bar{F}_V(c_n/x)}{x^2\,\bar{F}_V(c_n)}=H(\alpha)\frac{x^{\alpha-2}L(c_n/x)}{L(c_n)},$$

where $H(\alpha) := \dfrac{2}{\mathbb{E}\left((X\mathbf{1}_{\{X\geq 0\}})^\alpha\right)}\left(\dfrac{\alpha}{\alpha-2}+1\right)$. Hence, by Equations (4.10) and (4.11),

$$G_n(x) \leq \begin{cases} 2H(\alpha) & 0 < x \leq 1, \\ 2H(\alpha)\, x^{\alpha-1} & 1 < x \leq c_n/\bar{C}, \\ \frac{\mathbb{E}(W)}{c_n^2\, \bar{F}_V(c_n)} & x > c_n/\bar{C}. \end{cases} \tag{4.12}$$

By Equation (4.12) we obtain

$$\begin{aligned}
&\lim_{D\to\infty} \limsup_n \int_0^{c_n/\bar{C}} G_n(x)\, \mathbf{1}_{\{G_n(x)>D\}}\, \mathrm{N}(\mathrm{d}x) \\
&\qquad \leq \lim_{D\to\infty} \int_0^1 2H(\alpha)\, \mathbf{1}_{\{2H(\alpha)>D\}}\, \mathrm{N}(\mathrm{d}x) \\
&\qquad\quad + \lim_{D\to\infty} \int_1^\infty 2H(\alpha) x^{\alpha-1}\, \mathbf{1}_{\{2H(\alpha)x^{\alpha-1}>D\}}\, \mathrm{N}(\mathrm{d}x) = 0.
\end{aligned}$$

Finally,

$$\begin{aligned}
&\lim_{D\to\infty} \limsup_n \int_{c_n/\bar{C}}^\infty G_n(x)\, \mathbf{1}_{\{G_n(x)>D\}}\, \mathrm{N}(\mathrm{d}x) \\
&\qquad \leq \lim_{D\to\infty} \limsup_n \mathbf{1}_{\left\{\frac{\mathbb{E}(W)}{c_n^2\, \bar{F}_V(c_n)}>D\right\}} \frac{\mathbb{E}(W)}{c_n^2\, \bar{F}_V(c_n)} \left[1-\mathrm{N}(c_n/\bar{C})\right] \\
&\qquad \leq \limsup_n \frac{\mathbb{E}(W)}{c_n^2\, \bar{F}_V(c_n)} \left[1-\mathrm{N}(c_n/\bar{C})\right] \\
&\qquad = \limsup_n \frac{\mathbb{E}(W)\, \bar{\mathrm{N}}(c_n/\bar{C})\, e^{\lambda c_n}}{c_n^2\, \bar{F}_V(c_n)\, e^{\lambda c_n}} = 0,
\end{aligned}$$

where $\lambda > 0$, and where the last step follows, since the numerator vanishes in the limit, while the denominator tends to ∞, see Lemma A.7. This completes the proof of uniform integrability, that is, we have established Equation (4.7).

It remains to evaluate $\lim_{n\to\infty} G_n(X)$. For any $x > 0$,

$$\begin{aligned}
\lim_{n\to\infty} G_n(x) &= \lim_{n\to\infty} \frac{\int_{c_n/x} w^2\, F_{\sqrt{W}}(\mathrm{d}w)}{c_n^2 \bar{F}_V(c_n)} \\
&= \lim_{n\to\infty} \left[\left(\frac{\int_{c_n/x}^\infty w^2\, F_{\sqrt{W}}(\mathrm{d}w)}{(c_n/x)^2\, \bar{F}_{\sqrt{W}}(c_n/x)}\right) \Big/ \left(\frac{c_n^2\, \bar{F}_V(c_n)}{(c_n/x)^2\, \bar{F}_{\sqrt{W}}(c_n/x)}\right)\right].
\end{aligned}$$

By Lemma 4.7, we obtain for the limit of the numerator,

$$\lim_{n\to\infty} \left(\frac{\int_{c_n/x}^\infty w^2\, F_{\sqrt{W}}(\mathrm{d}w)}{(c_n/x)^2\, \bar{F}_{\sqrt{W}}(c_n/x)}\right) = \frac{\alpha}{\alpha-2}.$$

For the denominator, application of Lemma 4.4 and the definition of regularly varying functions yield

$$\begin{aligned}
\lim_{n\to\infty} \frac{c_n^2 \bar{F}_V(c_n)}{(c_n/x)^2\, \bar{F}_{\sqrt{W}}(c_n/x)} &= \lim_{n\to\infty} \left[\frac{\bar{F}_V(c_n)}{\bar{F}_{\sqrt{W}}(c_n)} \cdot \frac{x^2\, \bar{F}_{\sqrt{W}}(c_n)}{\bar{F}_{\sqrt{W}}(c_n/x)}\right] \\
&= \mathbb{E}\left((X\mathbf{1}_{\{X\geq 0\}})^\alpha\right) \cdot x^{-(\alpha-2)}.
\end{aligned}$$

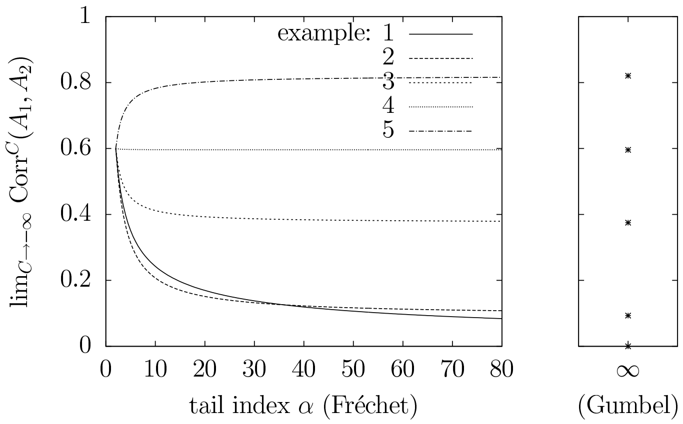

Figure 4: Asymptotic asset correlations as $C \to -\infty$.

Hence,

$$\lim_{n\to\infty} G_n(X) = \frac{\alpha \left(X\mathbf{1}_{\{X\geq 0\}}\right)^{\alpha-2}}{(\alpha-2)\,\mathbb{E}\left((X\mathbf{1}_{\{X\geq 0\}})^{\alpha}\right)} \quad \mathbf{P}\text{–a.s..}$$

Finally, taking expectation (continuing Equation (4.6)), we obtain

$$\mathbb{E}\left[\lim_{n\to\infty} G_n(X)\right] = \frac{\alpha\,\mathbb{E}\left((X\mathbf{1}_{\{X\geq 0\}})^{\alpha-2}\right)}{(\alpha-2)\mathbb{E}\left((X\mathbf{1}_{\{X\geq 0\}})^{\alpha}\right)} = \frac{\alpha}{(\alpha-1)(\alpha-2)}, \tag{4.13}$$

where the last equality follows, because by partial integration

$$\int_0^\infty x^a\,\phi(x)\,\mathrm{d}x = \frac{1}{a+1}\int_0^\infty x^{a+2}\,\phi(x)\,\mathrm{d}x, \quad a > 0.$$

Since Equation (4.13) holds for any sequence $(c_n)_{n\geq 1}$, we have established Equation (4.5). Combining Equations (4.4) and (4.5) completes the proof. □

4.3 V in the Gumbel MDA

Let us now consider the case when V is in the MDA of the Gumbel distribution. The MDA of the Gumbel distribution comprises a wide range of distribution functions, such as the lognormal, exponential, and the normal distributions.

The following proposition is the main result of this section.

Proposition 4.8. *Let $V \in MDA(\Lambda)$. Then,* $\lim_{C\to-\infty} \dfrac{\mathrm{Var}^C(V)}{\mathbb{E}^C(W)} = 0.$

The corresponding formula for asset correlations is obtained by inserting this expression into Equation (3.1).

Corollary 4.9. *Let $V \in MDA(\Lambda)$. Then, for $|\rho_i| < 1$ and $|\rho_j| < 1$,*

$$\lim_{C\to-\infty} \mathrm{Corr}^C(A_i, A_j) = \frac{\rho_{ij} - \rho_i\,\rho_j}{\sqrt{(1-\rho_i^2)(1-\rho_j^2)}} = \mathrm{Corr}(Z_i, Z_j). \tag{4.14}$$

If either $|\rho_i| = 1$ or $|\rho_j| = 1$ (but not both), then $\lim_{C\to-\infty} \mathrm{Corr}^C(A_i, A_j) = 0$. *If $\rho_i = \rho_j = \pm 1$, then* $\lim_{C\to-\infty} \mathrm{Corr}^C(A_i, A_j) = 1$. *If $\rho_i = -\rho_j = \pm 1$, then* $\lim_{C\to-\infty} \mathrm{Corr}^C(A_i, A_J) = -1$.

The asymptotic correlations for the examples of Table 1 are given in Figure 4.

For the proof of Proposition 4.8 we need the following properties of the tail function $\bar{F}_V(x)$ of V. Observe that from the first part of Lemma 4.7 one obtains a "mean excess function for higher moments", in the sense that

$$\mathbb{E}(V^k - x^k | V > x) = k \int_x^\infty \frac{y^{k-1}\,\bar{F}_V(y)}{\bar{F}_V(x)}\,\mathrm{d}y, \quad x \geq 0. \tag{4.15}$$

Furthermore, it follows from Theorem A.9 that for $V \in \mathrm{MDA}(\Lambda)$

$$\lim_{C\to\infty} \frac{\int_C^\infty \bar{F}_V(x)\,\mathrm{d}x}{C\bar{F}_V(C)} = 0, \tag{4.16}$$

$$\lim_{C\to\infty} \int_C^\infty x\bar{F}_V(x)\,\mathrm{d}x = 0. \tag{4.17}$$

Lemma 4.10. *Let* $V \in MDA(\Lambda)$. *Then* $\lim_{C\to\infty} \frac{\int_C^\infty C\,\bar{F}_V(x)\,\mathrm{d}x}{\int_C^\infty x\,\bar{F}_V(x)\,\mathrm{d}x} = 1.$

Proof. The denominator vanishes, see Equation (4.17). Furthermore, the numerator is strictly smaller than the denominator, so it vanishes, too. By Lemma 4.1, $\bar{F}_V$ is continuous and therefore

$$\frac{\mathrm{d}}{\mathrm{d}C} \int_C^\infty x\,\bar{F}_V(x)\,\mathrm{d}x = -C\bar{F}_V(C) \neq 0, \quad C > 0,$$

so that the conditions for applying the rule of L'Hospital to $\frac{\int_C^\infty C\,\bar{F}_V(x)\,\mathrm{d}x}{\int_C^\infty x\,\bar{F}_V(x)\,\mathrm{d}x}$ are satisfied:

$$\lim_{C\to\infty} \frac{\int_C^\infty C\,\bar{F}_V(x)\,\mathrm{d}x}{\int_C^\infty x\,\bar{F}_V(x)\,\mathrm{d}x} = \lim_{C\to\infty} \frac{\int_C^\infty \bar{F}(x)\,\mathrm{d}x - C\bar{F}_V(C)}{-C\bar{F}_V(C)} = \lim_{C\to\infty} 1 - \frac{\int_C^\infty \bar{F}_V(x)\,\mathrm{d}x}{C\bar{F}_V(C)}.$$

But the last term vanishes, see Equation (4.16). □

Proof of Proposition 4.8. By symmetry of V, we may equally show that

$$\lim_{C\to\infty} \frac{\mathrm{Var}(V|V \geq C)}{\mathbb{E}(W|V \geq C)} = 0. \tag{4.18}$$

Let $C > 0$ and write $\mathrm{Var}(V|V \geq C)$ in the form

$$\begin{aligned}\mathrm{Var}(V|V \geq C) &= C^2 + \mathbb{E}(V^2 - C^2|V \geq C) - (C + \mathbb{E}(V - C|V \geq C))^2 \\ &= \mathbb{E}(V^2 - C^2|V \geq C) - 2C\,\mathbb{E}(V - C|V \geq C) - (\mathbb{E}(V - C|V \geq C))^2\,.\end{aligned}$$

We divide each component by $\mathbb{E}(V^2 - C^2|V \geq C)$ and analyse the limit for $C \to \infty$: by Equation (4.15) and Lemma 4.10,

$$\lim_{C\to\infty} \frac{2C\,\mathbb{E}(V - C|V \geq C)}{\mathbb{E}(V^2 - C^2|V \geq C)} = \lim_{C\to\infty} \frac{\int_C^\infty C\,\bar{F}_V(x)\,\mathrm{d}x}{\int_C^\infty x\,\bar{F}_V(x)\,\mathrm{d}x} = 1.$$

Furthermore, as $C \to \infty$

$$\frac{(\mathbb{E}(V - C|V \geq C))^2}{\mathbb{E}(V^2 - C^2|V \geq C)} = \frac{\mathbb{E}(V - C|V \geq C)}{2C} \cdot \frac{2C\,\mathbb{E}(V - C|V \geq C)}{\mathbb{E}(V^2 - C^2|V \geq C)} \to 0,$$

since $\dfrac{E(V-C|V\geq C)}{2C} = \dfrac{\int_C^\infty \bar F_V(x)\,\mathrm{d}x}{2C\bar F_V(C)}$ vanishes, cf. Equation (4.16).

By combining the equations above we obtain

$$\lim_{C\to\infty} \frac{\mathrm{Var}(V|V\geq C)}{\mathbb{E}(V^2-C^2|V\geq C)} = 0.$$

For completing the proof of Equation (4.18) we show that

$$\mathbb{E}(V^2-C^2|V\geq C) \sim 2\mathbb{E}(W|V\geq C). \tag{4.19}$$

From $\mathbb{E}(W\mathbf{1}_{\{V\geq C\}}) = \mathbb{E}\left(\mathbb{E}(W\mathbf{1}_{\{\sqrt{W}X\geq C\}}|W)\right) = \mathbb{E}\left(W\,\bar{\mathrm{N}}\left(\frac{C}{\sqrt{W}}\right)\right)$ we obtain

$$\frac{\mathbb{E}(V^2-C^2|V\geq C)}{2\mathbb{E}(W|V\geq C)} = \frac{\int_C^\infty x\,\bar F_V(x)\,\mathrm{d}x}{\mathbb{E}(W\mathbf{1}_{\{V\geq C\}})} = \frac{\int_C^\infty x\,\bar F_V(x)\,\mathrm{d}x}{\mathbb{E}\left(W\,\bar{\mathrm{N}}\left(\frac{C}{\sqrt{W}}\right)\right)}.$$

Both the numerator and the denominator vanish as $C\to\infty$; for the numerator this follows from Equation (4.17), and for the denominator this follows by Dominated Convergence. We apply the rule of L'Hospital (where the derivative of the denominator is obtained by applying the Dominated Convergence Theorem) and further the identity (derived by partial integration)

$$\phi(x) = \int_x^\infty y\,\phi(y)\,\mathrm{d}y = x\,\bar{\mathrm{N}}(x) + \int_x^\infty \bar{\mathrm{N}}(y)\,\mathrm{d}y,$$

and obtain

$$\frac{C\,\bar F_V(C)}{\mathbb{E}\left(\sqrt{W}\,\phi\left(\frac{C}{\sqrt{W}}\right)\right)} = \frac{C\,\bar F_V(C)}{\mathbb{E}\left(C\,\bar{\mathrm{N}}\left(\frac{C}{\sqrt{W}}\right) + \sqrt{W}\int_{C/\sqrt{W}}^\infty \bar{\mathrm{N}}(y)\,\mathrm{d}y\right)}.$$

Finally, it follows from $\bar F_V(z) = \mathbb{E}(\bar{\mathrm{N}}(z/\sqrt{W}))$, $z\in\mathbb{R}$, that

$$\frac{C\,\bar F_V(C)}{\mathbb{E}\left(C\,\bar{\mathrm{N}}\left(\frac{C}{\sqrt{W}}\right) + \sqrt{W}\int_{C/\sqrt{W}}^\infty \bar{\mathrm{N}}(y)\,\mathrm{d}y\right)} = \frac{C\,\bar F_V(C)}{C\,\bar F_V(C) + \int_C^\infty \bar F_V(y)\,\mathrm{d}y} \to 1,$$

as $C\to\infty$, where the limit is a consequence of Equation (4.16). This establishes Equation (4.19) and completes the proof. □

Remark 4.11. Since we have used only properties of the class of rapidly varying functions, to which the distribution functions with infinite right endpoint in the Gumbel MDA belong, Proposition 4.8 can be generalised to "If $\bar F_V$ is rapidly varying, then $\lim_{C\to-\infty} \dfrac{\mathrm{Var}^C(V)}{\mathbb{E}^C(W)} = 0$.".

5 Example: Application to stress testing

Let us demonstrate how the results from the previous sections can be applied in practice. The initial motivation for analyzing correlations under stress is to better understand the results of stress tests performed in financial institutions. A typical example of a credit portfolio stress test is presented in Bonti et al. (2006), where a sample investment banking portfolio consisting of 25,000 loans is modelled in a structural credit portfolio model, see Merton (1974). In the Merton model, firm i defaults if its asset value A_i (at some fixed time horizon) falls below a threshold $D_i\in\mathbb{R}$, chosen such that $\mathbf{P}(A_i\leq D_i)$ equals a given probability of default. Bonti et al. (2006)

specify the dependence structure of the asset variables via 75 systematic factors that represent geographic regions and industries. The factors are assumed to be normally distributed, which is still industry standard in credit portfolio modelling. The stress scenario is implemented by truncating one of the systematic factors, the risk factor for the automotive industry.

The results of the stress test reported in Bonti et al. (2006) show a pattern that is often encountered: the relative impact of the stress on the expected loss of the portfolio, i.e., the mean of the portfolio loss distribution, is much stronger than on the value-at-risk of the portfolio. In the specific stress test, the portfolio EL increased by 56% whereas the 99.98% VaR increased by 19% only.

The correlation formulas derived in the present paper provide insight into the behaviour of these risk measures under stress. First of all, note that the systematic country and industry factors typically show high positive (unstressed) correlations, which induce positive correlations of systematic factors and asset variables of the individual counterparties. In a normally distributed model, these correlations are significantly reduced under stress as illustrated in Figure 2. It is therefore not surprising that the increase of the stressed VaR, which strongly depends on the asset correlations in the stressed model, is less significant than the increase of the stressed EL, which only depends on exposure and stressed default probabilities.

In addition to the analysis of stress tests, the formulas for correlations under stress can be used to design new meaningful stress scenarios, where the conditional correlation is set to an appropriate value, such as an empirical conditional correlation. We provide two examples of credit portfolio stress tests for illustration.

Both stress tests are implemented in a structural credit portfolio model with NVM-distributed asset variables as specified in (2.1) and (2.2). We make two simplifying assumptions, which, however, do not affect the qualitative behaviour of the results: First, we assume that the correlations of X and Y_i and the correlations of Y_i and Y_j are homogeneous, i.e., there exist $\rho, \bar{\rho}$ such that

$$\rho = \mathrm{Corr}(X, Y_i), \qquad \bar{\rho}^2 = \mathrm{Corr}(Y_i, Y_j) \tag{5.1}$$

for all $i, j \in \{1, \ldots, k\}$, $i \neq j$. In this case, the asset variables $A_i := \sqrt{W} Y_i$, $i = 1, \ldots, k$, can be conveniently represented by the systematic factor $V = \sqrt{W} X$, an additional systematic factor $\sqrt{W} Y$ and k specific factors $\sqrt{W}\varepsilon_1, \ldots, \sqrt{W}\varepsilon_k$, where $X, Y, \varepsilon_1, \ldots, \varepsilon_k$ are standard normally distributed and independent:

$$A_i = \sqrt{W}\left(\bar{\rho}\left(\frac{\rho}{\bar{\rho}} X + \sqrt{1 - \rho^2/\bar{\rho}^2}\, Y\right) + \sqrt{1 - \bar{\rho}^2}\,\varepsilon_i\right). \tag{5.2}$$

It is easily seen that $\mathrm{Corr}(\sqrt{W} X, A_i) = \mathrm{Corr}(X, Y_i)$ and that $\mathrm{Corr}(A_i, A_j) = \mathrm{Corr}(Y_i, Y_j)$.

As a second simplification, we consider only credit portfolios consisting of loans with identical default probability p and we assume that the portfolios are infinitely granular with equally weighted loans with total value 1 (in some monetary unit). In the normally distributed case, this leads to the well-known Vasicek formula, Vasicek (1991), which is the basis for the Basel 2 formula for regulatory capital for credit risk. The Vasicek formula is easily generalised to normal variance mixtures, in which case the portfolio loss is given by

$$L := \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \mathbf{1}_{\{A_i \leq D\}} = \mathrm{N}\left(\frac{D/\sqrt{W} - \bar{\rho}\left(\rho/\bar{\rho} X + \sqrt{1 - \rho^2/\bar{\rho}^2}\, Y\right)}{\sqrt{1 - \bar{\rho}^2}}\right) \quad \mathbf{P}\text{–a.s.}, \tag{5.3}$$

where N denotes the normal distribution function and D is the truncation threshold that yields a default probability p, i.e., $\mathbf{P}(A_i \leq D) = p$. Both the Vasicek formula and Equation (5.3) follow

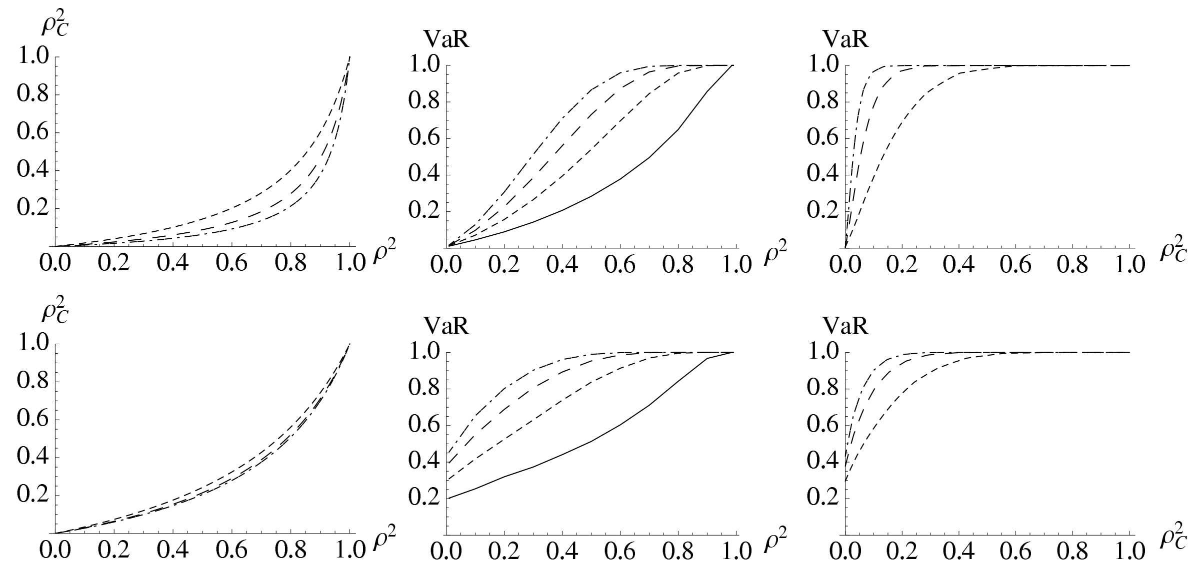

Figure 5: Top row: normally distributed asset returns; bottom row: t-distributed asset returns ($\nu = 5$). Left column: Unconditional correlation ρ^2 vs. conditional correlation ρ_C^2 for different stress probabilities; middle column: VaR for different stress probabilities; right column: VaR for different stress probabilities with correlation stressed. Stress probabilities: solid line: unstressed; small dashes: 10%; wide dashes: 1%; dash-dot-dash: 0.1%.

from the Strong Law of Large Numbers by conditioning on W, X and Y, and from Fubini's Theorem; for further details see e.g. (Bluhm et al. 2003, Proposition 2.5.4).

Note that Equation (5.3) illustrates another interpretation of the general NVM model: instead of NVM-distributed systematic and specific components with a common mixing variable, which introduces dependence between the two components, one can think of normally distributed independent components and a stochastic default threshold $D/\sqrt{W}$.

5.1 Stressing the correlation

In the first example, we choose a one-factor model with uncorrelated specific components, that is, the asset variables are given by Equation (5.2) with $\bar{\rho}^2 = \rho^2 = \mathrm{Corr}(A_i, A_j)$ for any two assets i, j. We set the default probability to $p = 0.005$, and we examine value-at-risk (VaR) at the 99.9% level for different correlations and stress levels. The truncation threshold C is chosen according to given stress probabilities of 10%, 1%, 0.1%, that is, $\mathbf{P}(V \leq C) \in \{10\%, 1\%, 0.1\%\}$. We consider both a normally distributed model and a t-distributed model with parameter $\nu = 5$.

In the middle column of Figure 5, the unstressed and stressed VaR's are shown as a function of the unstressed correlation ρ^2. As can be seen, VaR increases with the stress level and with ρ^2. Furthermore, VaR is greater in the t-distributed case than in the normally distributed case.

In these stress scenarios the dependence structure is specified via unstressed correlations. It is ignored that correlations under stress may behave differently than implied by the model. The formulas for conditional correlation derived in the previous sections can be applied to set the conditional correlation to a target value rather than relying on the values implied by the model. The left column of Figure 5 shows the mapping of asset correlation ρ^2 to the correlations under stress $\mathrm{Corr}^C(A_i, A_j) =: \rho_C^2$. This allows for choosing target correlations under stress by inserting appropriate unstressed correlations into the model. With the formulas for conditional correlation derived for normally and t-distributed random variables one can make the notion of a stressed correlation precise, whereas for the more general normal variance mixtures the asymptotic correlation may be used to derive the unstressed correlation.

Thus, in a second step, we consider the correlation itself as an additional risk factor subjected to stress. Here, we employ a "constant correlation" assumption in the sense that we require the target conditional correlation to equal the original unstressed correlation. The resulting VaR's under stress are shown in the right column of Figure 5. Since in our example conditional correlation decreases with increasing stress, these stressed VaR's are greater than the stressed VaR's in the first case. In particular, under high stress, VaR increases to 1, the notional amount of the portfolio, even for moderately low correlations. On the other hand, the differences in expected loss, which corresponds to the probability of default, between the two stress scenarios are relatively small compared to the differences in VaR.

The example gives an explanation for the observation of the relatively moderate increase of VaR under stress compared to EL mentioned at the beginning of the section. It confirms that decreasing conditional correlation in light-tailed models is a key driver of this behaviour. Moreover, we demonstrate that choosing a heavy-tailed distribution or that stressing the correlation in an appropriate way mitigates the problem.

All calculations were done by simulating the risk factor. For each given stress probability, 100,000 simulations fulfilling the condition $\{V \leq C\}$ were drawn. To assess the quality of the estimates we calculate the sample standard deviation of the VaR estimator assuming a correlation of 0.5 by simulating the estimator ten times. The resulting standard deviations are within 2% of the VaR figure in the normally distributed case and within 1.2% in the t-distributed case.

In the setup above, one may also consider one of the assets as a risk factor. Truncating the asset return at its default boundary and providing the correlation under stress may be useful for studying contagion effects.

5.2 Empirical example

In contrast to the first example we now implement stress scenarios in the unrestricted two-factor model (5.3) and calibrate model parameters to empirical data, consisting of rating data and equity time series of the DAX and its 30 constituents. The default probability p that enters the Vasicek formula is set to 0.058% (i.e., 5.8 basis points). This value is derived from S&P credit ratings of the DAX companies.[10] Taking the DAX return to be the systematic factor, the correlation parameters are calibrated to historical equity data, i.e., to daily returns from the DAX index and its $n = 30$ constituents ranging from 5 February 2001 to 20 December 2011.[11]

[10] The credit ratings are mapped to one-year default probabilities, using cumulative default rates of European corporates from 1981 until 2010, S&P (2011). The default probabilities are in turn mapped to hazard rates, of which a weighted average is calculated. From this, an average default probability of 0.00058 is determined. Three names do not have a credit rating, making up a total of 9.62% of the DAX, which we simply ignore in our calculations.

[11] Since the time series data are typically not iid, one can consider applying a GARCH filter to each time series. This entails fitting each time series to a GARCH model and dividing each return by its corresponding volatility estimate from the GARCH model, yielding the so-called innovations, which are assumed to be iid. Each innovation is then multiplied with the volatility forecast for the next time period. Under the assumption of iid innovations, this yields a sample of iid returns. We refer to e.g. (Barone-Adesi et al. 1998), (McNeil and Frey 2000), (Alexander and Sheedy 2008) for further details on GARCH filtering.

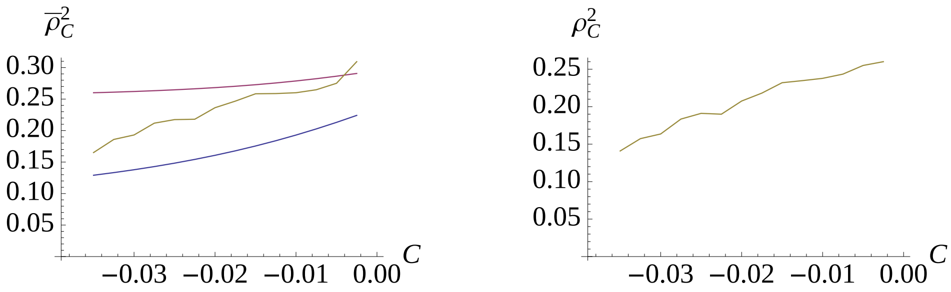

Figure 6: Average correlations under stress $\bar{\rho}_C^2 = \text{Corr}^C(A_i, A_j)$ (left), resp. $\rho_C^2 = (\text{Corr}^C(V, A_i))^2$ (right); solid line (small dashes): assuming constant ρ^2 and $\bar{\rho}^2$ and normally distributed (t-distributed) asset returns; wide dashes: $\bar{\rho}_C^2$, resp. ρ_C^2, implied by the data.

The average (unstressed) correlations are given by

$$\rho = \text{Corr}(V, A_i) = \frac{1}{n}\sum_{i=1}^{n} \rho_i = 0.6161$$

$$\bar{\rho}^2 = \text{Corr}(A_i, A_j) = \frac{1}{n^2 - n(n-1)/2}\sum_{i=1}^{n}\sum_{j=i+1}^{n} \rho_{ij} = 0.415.$$

Note that $\bar{\rho} = 0.6442$ and that $\rho^2 = 0.3796$.

We use the formulas for conditional correlation to calculate the correlations under stress for ρ^2 and $\bar{\rho}^2$ as estimated for both the normal distribution and the t-distribution, where in the latter case we set $\nu = 3.46$, which is the average tail index estimated from the DAX returns. In addition, we calculate average empirical correlations when the DAX is subjected to stress. As can be seen from Figure 6, the empirical conditional correlation behaves differently than the conditional correlations in both the normally distributed and the t-distributed case. For thresholds close to 0, the empirical conditional correlation is similar to the t-distributed case. For more severe stress the empirical conditional correlation is smaller than in the t-distributed case, but still greater than in the normally distributed case.[12] Of course, as the stress increases, the sample size decreases. The actual number of samples under conditioning ranges from 74 at $C = -0.035$ to 1069 at $C = -0.0025$ (the total number of returns is 2769).

We calculate VaR under stress for ρ^2 and $\bar{\rho}^2$ as estimated for both the normal distribution and the t-distribution. We repeat the calculations, but now with the empirical conditional correlations. For this we back out unstressed correlations from the empirical stressed correlations ρ_C^2 and $\bar{\rho}_C^2$ that need to be fed into the model to produce the target correlations under stress.

The resulting EL's and VaR's are shown in Figure 7.[13] In the normally distributed case the lower conditional correlation (compared to the empirical conditional correlation) translates into lower stressed EL's and VaR's (see top graphs of Figure 7). Thus, imposing the empirical

[12]Observe that one should not necessarily expect that conditional correlations increase with the level of stress applied. This so-called *correlation breakdown* refers to an increase of *dynamic* correlations in stressed markets and has been verified empirically in different markets, see for example Longin and Solnik (2001), Loretan and English (2000), Ang and Chen (2002), Ang and Bekaert (2002), Kim and Finger (2000). It has gained wide interest as it annihilates the risk reducing effect of a well-diversified or a hedging portfolio when it is needed most, namely in a stressed market. The association of conditional correlation with correlation breakdown is not straightforward: one cannot necessarily deduce from the behaviour of conditional correlations that correlations change with time or state. This point is discussed in detail in Boyer et al. (1999).

[13]All calculations are done by Monte Carlo simulation with 500,000 simulations fulfilling $\{\sqrt{W}X \leq C\}$ for each truncation level C.

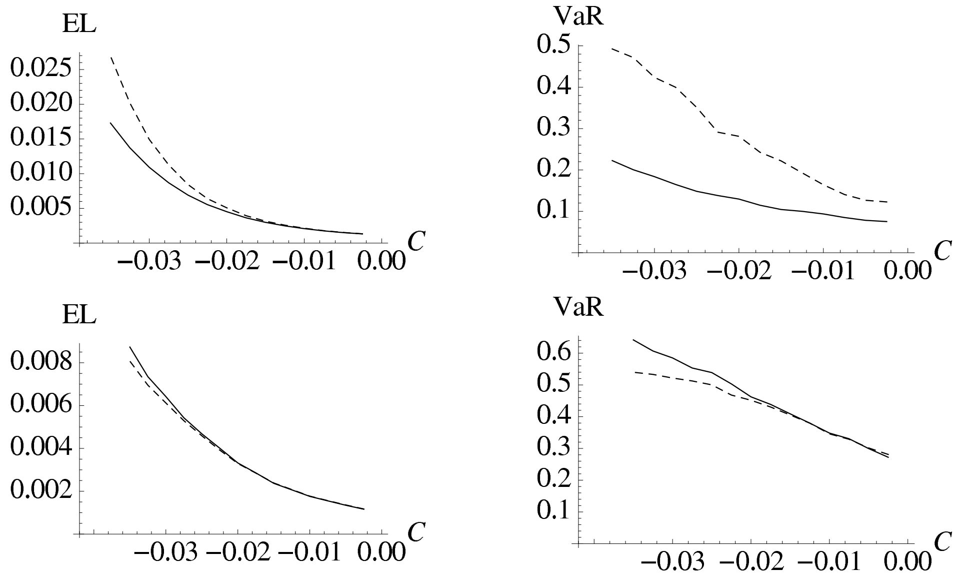

Figure 7: EL (left) and VaR (right) for the DAX portfolio under stress. Top: normal distribution case; bottom: t-distribution case. Solid line: Assuming constant ρ^2 and $\bar{\rho}^2$; dashed line: setting ρ_C^2 and $\bar{\rho}_C^2$ to the values determined from the data.

conditional correlations rather than using the unconditional estimates gives a more prudent estimate of risk under the stress scenario. For the t-distributed case, the impact of the correlation parameter turns out to be more moderate: in line with the correlation analysis in Figure 6, the stressed EL and VaR figures turn out to be similar for truncation levels greater than -2% (given the scarcity of data, a clear interpretation for lower truncation levels may be difficult). Hence, in this example the t-distributed model provides a consistent framework for modelling unstressed as well as realistic stressed correlations.

6 Conclusion

We study correlations in truncated NVM distributed models. The main motivation for our analysis comes from stress testing. Stress tests on bank portfolios have gained particular importance as a technique for assessing the adequacy of capital buffers in financial institutions, and as such form an integral part of risk management and banking supervision.

The class of NVM models is significant for practical applications as it encompasses many models commonly encountered in the financial industry. For example, our results can be applied to the most popular credit portfolio models such as CreditMetrics™ and Moody's KMV Portfolio Manager™, where log-returns of asset values follow a multivariate normal distribution. The appeal of the NVM distribution in the industry certainly comes from its simplicity combined with diverse tail behaviour, ranging from light- to heavy-tailed.

From a technical point of view, our results are summarised as follows: for the special cases of jointly normally and t-distributed asset returns we have derived closed formulas for the correlation in the truncated model. More generally, for the NVM distribution, we have calculated the asymptotic limit of the correlation, which depends on whether the variables are in the maximum domain of attraction of the Fréchet or Gumbel distribution. It turns out that correlations in heavy-tailed NVM models are less sensitive to stress than in medium- or light-tailed models.

Our analysis allows for a critical review ot the suitability of this model class to reproduce empirically observed conditional correlations, and as such provides valuable information for risk and capital management in financial institutions. Also, using our results, the conditional correlation can take the role of a further free parameter, which can be set to a target value, for example to design more prudent stress tests.

We illustrate the application of our correlation formulas by comparing value-at-risk under stress for a stylized credit portfolio, both under different distribution and correlation assumptions, and using empirical conditional correlations derived from DAX data.

A Some results from extreme value theory

We give a brief review of some results of extreme value theory that are applied in this paper; standard references to EVT are Embrechts et al. (1997), Resnick (2007) and de Haan and Ferreira (2006).

The central result of classical extreme value theory is the Fisher-Tippett Theorem, which specifies the form of the limit distribution for centered and normalized maxima.

Theorem A.1 (Fisher-Tippett Theorem, limit laws for maxima)**.** *Let* (X_n) *be a sequence of iid random variables, and let* $M_n = \max(X_1, \ldots, X_n)$*. If there exist norming constants* $c_n > 0$*,* $d_n \in \mathbb{R}$ *and some non-degenerate distribution function* H *such that*

$$\frac{M_n - d_n}{c_n} \overset{\mathcal{L}}{\to} H, \qquad \text{as } n \to \infty, \tag{A.1}$$

then H *belongs to the type of one of the following three distribution functions:*

$$\text{Fréchet:} \qquad \Phi_\alpha(x) = \begin{cases} 0, & x \le 0 \\ \exp\{-x^{-\alpha}\}, & x > 0 \end{cases} \qquad \alpha > 0.$$

$$\text{Weibull:} \qquad \Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^{\alpha}\}, & x \le 0 \\ 1, & x > 0 \end{cases} \qquad \alpha > 0.$$

$$\text{Gumbel:} \qquad \Lambda(x) = \exp\{-e^{-x}\}, \quad x \in \mathbb{R}.$$

If Equation (A.1) holds, then the type of the limiting distribution is uniquely determined, and does not depend on the particular choice of norming constants c_n, d_n.

Definition A.2. A random variable X with distribution function F belongs to the *maximum domain of attraction of H* if there exist constants $c_n > 0$, $d_n \in \mathbb{R}$ such that Equation (A.1) holds, written $X \in \mathrm{MDA}(H)$ and $F \in \mathrm{MDA}(H)$.

A different parameterization allows grouping the Fréchet, Weibull and Gumbel distributions into one family of distribution functions, called the *generalised extreme value distributions.*

A.1 Maximum domain of attraction of the Fréchet distribution

The distributions that are in $\mathrm{MDA}(\Phi_\alpha)$, $\alpha > 0$, have a particularly elegant and convenient representation via slowly varying functions or regularly varying functions.

Definition A.3. A positive, Lebesgue-measurable function L on $(0, \infty)$ is *slowly varying (at ∞)* if

$$\lim_{x \to \infty} \frac{L(tx)}{L(x)} = 1, \qquad t > 0.$$

A positive, Lebesgue-measurable function h on $(0, \infty)$ is *regularly varying (at ∞) with index $\rho \in \mathbb{R}$* if

$$\lim_{x\to\infty} \frac{h(tx)}{h(x)} = t^{\rho}, \qquad t > 0.$$

We shall drop the specification "at ∞", since we only consider functions that are regularly or slowly varying at ∞. It is easily seen that h is regularly varying with index ρ if and only if there exists a slowly varying function L such that $h(x) = x^{\rho} L(x)$ (just choose $L(x) = h(x)/x^{\rho}$). For details on regularly varying functions, we refer to Bingham et al. (1987).

The following theorem completely characterises those distribution functions that are in the MDA of the Fréchet distribution.

Theorem A.4 (Gnedenko (1943)). *The distribution function F belongs to MDA(Φ_α), $\alpha > 0$, if and only if $\bar{F}(x) := 1 - F(x) = x^{-\alpha} L(x)$ for some slowly varying function L.*

It follows directly that these distribution functions have infinite right endpoint, that is $\sup\{x : F(x) < 1\} = \infty$. Examples of distributions in MDA(Φ_α) are the t-distribution and the inverse gamma distribution. The parameter α is called the *tail index*. In the case of the t-distribution the degree of freedom is α.

An important result for regularly varying functions, which is used in Section 4.2, is Karamata's Theorem, see (Bingham et al. 1987, Theorem 1.5.11 (ii)) or (McNeil et al. 2005, Theorem A.5).

Theorem A.5 (Karamata's Theorem). *Let L be a slowly varying function that is locally bounded in $[x_0, \infty)$, for some $x_0 \geq 0$. Then,*

(i) for $\kappa > -1$,

$$\int_{x_0}^{x} t^{\kappa} L(t)\,\mathrm{d}t \sim \frac{1}{\kappa+1} x^{\kappa+1} L(x), \qquad \textit{as } x \to \infty,$$

(ii) for $\kappa < -1$,

$$\int_{x}^{\infty} t^{\kappa} L(t)\,\mathrm{d}t \sim -\frac{1}{\kappa+1} x^{\kappa+1} L(x), \qquad \textit{as } x \to \infty.$$

The following bound for regularly varying functions can be found e.g. in Potter (1942) and (de Haan and Ferreira 2006, Proposition B.1.9):

Theorem A.6 (Potter bound). *Suppose h is regularly varying with index ρ. For arbitrary $\varepsilon > 0$ and $\delta > 0$ there exists $x_0 = x_0(\varepsilon, \delta)$ such that for $x \geq x_0$, $x\,t \geq x_0$,*

$$(1-\varepsilon)t^{\rho} \min(t^{\delta}, t^{-\delta}) < \frac{h(tx)}{h(x)} < (1+\varepsilon)t^{\rho} \max(t^{\delta}, t^{-\delta}).$$

Conversely, if h satisfies the property above then h is regularly varying with index ρ.

The following is a direct consequence of Corollary 1.3.2 and Lemma 1.3.5 of (Embrechts et al. 1997):

Lemma A.7. *If the tail distribution function $\bar{F}$ is regularly varying, then $\lim_{x\to\infty} e^{\varepsilon x} \bar{F}(x) = \infty$, for all $\varepsilon > 0$.*

A.2 Maximum domain of attraction of the Weibull distribution

Distributions in MDA(Ψ_α), $\alpha > 0$, all have finite right endpoint. They include, for example, the uniform and the beta distributions.

A.3 Maximum domain of attraction of the Gumbel distribution

The MDA of the Gumbel distribution covers a wide range of distribution functions, such as the lognormal, exponential and the normal distributions. This class contains distribution functions with both finite and infinite right endpoint. If the distribution function $F \in \mathrm{MDA}(\Lambda)$ has an infinite right endpoint, then $\bar{F}$ is rapidly varying.

Definition A.8. A positive, Lebesgue-measurable function h on $(0, \infty)$ is *rapidly varying with index* $-\infty$ if

$$\lim_{x \to \infty} \frac{h(tx)}{h(x)} = \begin{cases} 0, & t > 1, \\ \infty, & 0 < t < 1. \end{cases}$$

For rapidly varying functions, there exists an analogous result to Karamata's Theorem, see e.g. (Embrechts et al. 1997, Theorem A3.12):

Theorem A.9. *Let h be a non-increasing, rapidly varying function. Then for some $z > 0$ and all $\kappa \in \mathbb{R}$*

$$\int_z^\infty t^\kappa \, h(t) \, \mathrm{d}t < \infty \qquad \textit{and} \qquad \lim_{x \to \infty} \frac{x^{\kappa+1} \, h(x)}{\int_x^\infty t^\kappa \, h(t) \, \mathrm{d}t} = \infty.$$

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